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# Fractional-order models of time delay systems using Walsh operational matrices

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***Abstract*— Fractional-order modeling is recently attracting the attention of researchers in many disciplines of science and engineering. In this paper, a simplified way to identify systems is proposed using Walsh functions. The complex procedure for system approximation in fractional-order domain has been reduced by converting the complex fractional calculus equations into simple algebra and therefore it allows the estimation of the implicit time-delay in the system together with other model parameters. The orthogonal Walsh operational matrix is employed for the assessment of the unknown system transfer function. The presented method does not require any prior knowledge of the transfer function structure or partial information about fractional differentiation order and facilitates low order model using time response data without and with considering the influence of noise. Numerical analysis and practical study on DC motor speed control system show the efficacy of the presented approach without additional filtering or signal smoothing and an extensive computation burden.**

## I. INTRODUCTION

The concept of fractional calculus (FC) has become more interesting and easily applicable with the advent of readily available powerful computers. Among its recent applications in signal processing, it has provided a means to tackle previously intractable problems related to complex dynamics representation [1]. Using fractional-order derivatives, we can observe more realistic behavior for real-world phenomena and physical systems. However, it is not easy to deal with fractional derivatives of input and output signals. Most classical integer-order (IO) methods for modeling cannot be directly applied for the estimation of a fractional-order model (FOM).

Fractional order modeling has been used for various applications. Modeling of supercapacitor [2], fractional filter design [3], fractional oscillators [4], servo system [5], lithium-ion batteries [6], diffusion process [7], viscoelasticity [8] have been well explained using FOMs.

So far, various techniques have been used to address the problem to identify the fractional order of the operators. The simple approach is to use the operational matrix into the system equation. This approach converts fractional derivative of signal into the matrix-vector form so the problem of complex FC calculation can be avoided.

Frequency domain techniques have been proposed in [9] and [10] for the estimation of system parameters and time delay. Narang et al. [11] proposed a linear filter based identification method. Victor et al. [12] approximated non-integer order and other parameters of FOS. They used simplified refined instrumental variable method with gradient based algorithm. A time-domain identification method using adjustable fractional order differentiator based on recursive least squares algorithm is illustrated in [13]. Ahmed [14] presented identification method using step response data assuming that the fractional orders were known. Nie et al. [15] proposed techniques using three points data (TPD) on the step response and single-variable search (SVS). More recently, block pulse functions were used by Tang et al. [16] to make block pulse operational matrices (BPOM) for FOS with time delay.

Although the orthogonal basis functions are becoming very popular in fractional-order identification, the challenge is yet to address when the system has a time delay. Most studied so far for fractional order identification are either only those with no time delays or those that require some initial information to estimate the parameters of the system model. In most reported methods, the time delay estimation is treated separately from the estimation of model parameters. So, how to identify the FOS more accurately with less prior information is still an open problem.

This article proposes an approach, via the Walsh functions, to estimate both model and time delay parameters together from output signal data generated after a single step or random input sequence signal. The Walsh operational matrix (WOM) can be obtained using Walsh orthogonal basis. It can convert the complex integral-differential equations of any fractional orders into simple algebraic equations. Without having prior knowledge of order or structure of the transfer function, fractional first order model can be identified based on input-output data. It has given overall more simplicity and accurate estimation without much computational complexity.

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## II. Basic concepts

### A. Definition of fractional calculus

Fractional calculus is a generalization of non-integer (real) order integration and differentiation and its operator is generally defined as

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \alpha > 0; \\ 1 & \alpha = 0; \\ \int\limits_a^t (d\tau)^{-\alpha} & \alpha < 0; \end{cases} \tag{1} $$

where $a$ and $t$ are the bounds of the operation and $\alpha$ ($\alpha \in R$) is the order of operation. In our work, we use the Riemann-Liouville (R-L) definition given by

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int\limits_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{2} $$

where $n-1 < \alpha < n$ , $n \in N$, and $\Gamma$ denotes gamma function.

Suppose the FO derivative and integral are represented in the Laplace domain with zero initial. Then [17],

$$ L\left[{}_0D_t^\alpha f(t)\right] = s^\alpha F(s) \tag{3} $$

$$ L\left[I_0^\alpha f(t)\right] = \frac{1}{s^\alpha} F(s) \tag{4} $$

where $s^\alpha$ is a fractional Laplacian operator.

### B. Walsh functions

Basically, the Walsh functions are the complete set of orthogonal basis functions developed by J. Walsh in 1923 [18]. From technical point of view, they have similar properties with the classical trigonometric *sine* and *cosine* sets of orthogonal functions. Therefore, each function is either even or odd with respect to half of the interval, with neither function vanishing identically on any sub-interval. However, they take only the values 1 and $-1$ which make them simpler than trigonometric functions [19].

Consider an arbitrary signal $x(t)$ which is absolutely integrable over an interval $[0, 1)$ and so can be expanded into Walsh series as [20],

$$ x(t) = \sum_{i=0}^{M} c_i w_i(t) \approx C_M^T W_M(t) \tag{5} $$

where, $c_i = \int\limits_0^1 w_i(t)x(t)\mathrm{dt}$ is the Walsh coefficient, $C_M \triangleq [c_0, c_1, \ldots, c_{M-1}]^T$ is the Walsh coefficient vector and $W_M(t) \triangleq [w_0(t), w_1(t), \ldots, w_{M-1}(t)]^T$ is the Walsh function vector. It is preferable to take the dimension number $M$ as a power of 2.

After applying a linear transformation between block-pulse and Walsh functions, one can derive another form of Walsh functions as [20]

$$ W_M(t) \approx \Omega_{M\times M} \psi_M(t) \tag{6} $$

where $\psi_M(t)$ represents block pulse functions and $\Omega_{M\times M}$ is an $M$-square Walsh matrix.

## III. Walsh operational matrix for fractional order integration

Walsh functions are piecewise constant functions and analytical expression for WOM of integration can be derived by using operational matrix of integration of block pulse functions [16]. The integration of block pulse functions can be written as,

$$ (I^\alpha \psi_M)(t) \approx F_\alpha \psi_M(t) \tag{7} $$

where $F_\alpha$ is the $M$-square generalized operational matrix of fractional order integration (FOI) and $\alpha > 0$ is the real number. Suppose, the fractional integration of the Walsh function $W_M(t)$ is expressed as above then,

$$ \left(I^\alpha W_M\right)(t) \approx P^\alpha_{M\times M} W_M(t) \tag{8} $$

where $P^\alpha_{M\times M}$ is defined as a WOM. With the use of (7) in (6), one can get the integration of $W_M(t)$ as

$$ \left(I^\alpha W_M\right)(t) \approx \Omega_{M\times M} I^\alpha \psi_M(t) \approx \Omega_{M\times M} F_\alpha \psi_M(t) \tag{9} $$

From (8) and (9), we write the simplified expression of WOM for FOI as

$$ P^\alpha_{M\times M} \approx \Omega_{M\times M} F_\alpha \Omega^{-1}_{M\times M} \tag{10} $$

## IV. Delay operational matrix for Walsh functions

It is to mention that the operational matrix of FOI will be different when a time delay $\theta$ appears into any absolutely integrable function. In order to design the operational matrix of the delayed Walsh function, the shifted block pulse functions $\psi_M(t-\theta)$ can be taken as

$$ \psi_M(t-\theta) = E\psi_M(t) \tag{11} $$

where $E$ is the $M$-square generalized delay operational matrix [16]. Similar ways, we can take the delayed Walsh function $W_M(t-\theta)$ as in (11) and write

$$ W_M(t-\theta) = ZW_M(t) \tag{12} $$

where $Z$ is the $M$-square delay operational matrix of Walsh functions. Using (6), $W_M(t-\theta)$ can be expressed as

$$ W_M(t-\theta) \approx \Omega_{M\times M} \psi_M(t-\theta). \tag{13} $$

Here, (13) states time delayed Walsh function on the interval [0, $t_f$] and $t > \theta$. Now from (12) and (13), we get

$$ZW_M(t) \approx \Omega_{M\times M}\psi_M(t-\theta) \tag{14}$$

Substituting (6) and (11) into (14), one obtains

$$Z\Omega_{M\times M}\psi_M(t) \approx \Omega_{M\times M}E\psi_M(t) \tag{15}$$

Finally, the Walsh delay operational matrix $Z$ can be obtained as

$$Z \approx \Omega_{M\times M}E\Omega^{-1}_{M\times M} \tag{16}$$

Through an algebraic rearrangement of the terms, the fractional integration on $W_M(t-\theta)$ can be rewritten as

$$(I^\alpha W_M)(t-\theta) \approx Z(I^\alpha W_M)(t) \approx ZP^\alpha_{M\times M}W_M(t) \tag{17}$$

Here, $(I^\alpha W_M)(t-\theta)$ can be obtained simply by matrix multiplication of $W_M(t)$ with $P^\alpha_{M\times M}$ and $Z$. Therefore, integration of function is obtained by algebraic matrix multiplication without calculus terms. It is worth noting here that the application of operational matrices omits the complex calculation of fractional-order equations and reduces complexity of system modeling.

## V. Proposed technique for fractional order model with input time delay

Consider a general continuous-time single input single output (SISO) fractional order system with time delay described by the following differential equation:

$$\sum_{i=0}^{n} a_i D^{\alpha_i} y(t) = \sum_{j=0}^{m} b_j D^{\beta_j} u(t-\theta) \tag{18}$$

where $a_i(i=0,\dots,n)$ and $b_j(j=0,\dots,m)$ are arbitrary real numbers, $\alpha_i(i=0,\dots,n)$ and $\beta_j(j=0,\dots,m)$ are fractional differential orders, $y(t)$ and $u(t)$ are the output and input of the system, respectively. Equation (18) can be expanded as

$$\begin{aligned} &a_n D^{\alpha_n} y(t) + a_{n-1}D^{\alpha_{n-1}}y(t) + \cdots + a_0 D^{\alpha_0}y(t) = \\ &b_m D^{\beta_m}u(t-\theta) + b_{m-1}D^{\beta_{m-1}}u(t-\theta) + \cdots + b_0 D^{\beta_0}u(t-\theta) \end{aligned} \tag{19}$$

If we integrate both sides of (19) by fractional-order $\alpha_n$, we can easily write

$$\begin{aligned} &a_n y(t) + a_{n-1}I^{\alpha_n-\alpha_{n-1}}y(t) + \cdots + a_0 I^{\alpha_n-\alpha_0}y(t) = \\ &b_m I^{\alpha_n-\beta_m}u(t-\theta) + b_{m-1}I^{\alpha_n-\beta_{m-1}}u(t-\theta) + \cdots + \\ &\qquad b_0 I^{\alpha_n-\beta_0}u(t-\theta) \end{aligned} \tag{20}$$

If the output and input of the system is expressed in terms of Walsh functions, then

$$\begin{aligned} y(t) &\approx Y^T W_M(t) \\ u(t-\theta) &\approx U^T W_M(t-\theta) \end{aligned} \tag{21}$$

where $Y^T = [y_1, y_2, \dots, y_M]$ and $U^T = [u_1, u_2, \dots, u_M]$. Using (8), (12) and (21), the following algebraic terms can be derived:

$$\begin{aligned} I^{\alpha_n-\alpha_{n-1}}y(t) &\approx Y^T I^{\alpha_n-\alpha_{n-1}}(W_M(t)) \\ &\approx Y^T P^{\alpha_n-\alpha_{n-1}}_{M\times M}W_M(t) \end{aligned} \tag{22}$$

and

$$\begin{aligned} I^{\alpha_n-\beta_{m-1}}u(t-\theta) &\approx U^T I^{\alpha_n-\beta_{m-1}}Z(W_M(t)) \\ &\approx U^T Z P^{\alpha_n-\beta_{m-1}}_{M\times M}W_M(t) \end{aligned} \tag{23}$$

Then (20) can be written as

$$\begin{aligned} &Y^T(a_n I + a_{n-1}P^{\alpha_n-\alpha_{n-1}}_{M\times M} + \cdots + a_0 P^{\alpha_n-\alpha_0}_{M\times M}) \approx \\ &U^T Z(b_m P^{\alpha_n-\beta_m}_{M\times M} + b_{m-1}P^{\alpha_n-\beta_{m-1}}_{M\times M} + \cdots + b_0 P^{\alpha_n-\beta_0}_{M\times M}) \end{aligned} \tag{24}$$

Finally, the output $y(t)$ can be calculated from (21) and (24) as

$$y(t) \approx U^T Z\left(\sum_{j=0}^{m} b_j P^{\alpha_n-\beta_j}_{M\times M}\right)\left(a_n I_{M\times M} + \sum_{i=0}^{n-1} a_i P^{\alpha_n-\alpha_i}_{M\times M}\right)^{-1} W_M(t) \tag{25}$$

(25) is an algebraic expression with matrices $Z$ and $P$ that contain the modeling parameters. It can be seen that the fractional differential equation is represented into a simple matrix algebraic equation.

## VI. System model and method validation

In order to validate the proposed technique, the procedure is estimating the single pole general transfer function model that frequently used in practice.

$$g(s) = b_0 e^{-\theta s}/(a_1 s^\alpha + a_0) \tag{26}$$

Note that the order $\alpha$ makes the transfer function in fractional domain, where $\alpha > 0$. The above transfer function represents the continuous-time fractional first order model with time delay. From (25), system output $y(t)$ can be represented for single pole model of (26) as,

$$y(t) \approx U^T Z\left(b_0 P^\alpha_{M\times M}\right)\left(a_1 I + a_0 P^\alpha_{M\times M}\right)^{-1} W_M(t) \tag{27}$$

To derive the open-loop transfer function of identified system, it is necessary to have available some appropriate datasets from the system. Given that the step response datasets $\{udata(k), ydata(k)\}_{k=1,\cdots M}$, the method estimates the unknown transfer function parameters. The proposed technique attempts to solve the problem:

$$ISTE = \min_\rho \sum\nolimits_{k=1}^{M} [k(y(k) - ydata(k))]^2, \tag{28}$$

where $\rho$ is the vector of unknown parameters $(b_0, a_0, a_1, \alpha, \theta)$, $y(k)$ is the time domain response of (27) calculated using $\rho$, $ydata$ is the collected step response to fit to $y(k)$. In (28) $y(k)$ and $ydata(k)$ are the simulated response and collected response data at time $t_k$, and $M$ is the total number of data points in

TABLE I

COMPARISON OF IDENTIFIED MODELS

| Systems | Methods | Identified models | $E_t \times 10^{-4}$ | $E_f(\%)$ |
|---|---|---|---|---|
| $g_1(s)$ | Proposed (w/n) | $\frac{0.9759}{0.9911s^{0.6188}+1.9615}e^{-0.4030s}$ | 0.075 | 1.182 |
| | BPOM [16] | $\frac{0.9519}{1.0320s^{0.5808}+1.9559s^{0.0240}}e^{-0.3924s}$ | 1.191 | 4.720 |
| $g_2(s)$ | Proposed (w/n) | $\frac{2.0500}{11.1409s^{1.3265}+2.0331}e^{-2.8312s}$ | 2.971 | 1.482 |
| | Proposed (n) | $\frac{1.8751}{10.0531s^{1.3230}+1.8615}e^{-2.6715s}$ | 1.578 | 1.291 |
| $g_3(s)$ | Proposed (w/n) | $\frac{1.0028}{4.2591s^{1.5545}+0.98223}e^{-0.3374s}$ | 9.935 | 2.763 |
| | Proposed (n) | $\frac{1.0163}{3.9965s^{1.54}+1.0143}e^{-0.9303s}$ | 18.550 | 4.375 |
| | TPD [15] | $\frac{1}{3.5294s^{1.5216}+1}e^{-0.6265s}$ | 22.216 | 2.787 |
| | SVS [15] | $\frac{1}{4.2556s^{1.58}+1}e^{-0.37336s}$ | 11.466 | 1.227 |
| $g_4(s)$ | Proposed (w/n) | $\frac{0.9478}{1.2122s^{1.2435}+0.9672}e^{-0.6378s}$ | 1.804 | 0.222 |
| | Proposed (n) | $\frac{0.9684}{1.0810s^{1.1485}+0.9559}e^{-0.8749s}$ | 10.132 | 3.157 |
| | TPD [15] | $\frac{1}{1.2229s^{1.2417}+1}e^{-0.6786s}$ | 4.434 | 2.967 |
| | SVS [15] | $\frac{1}{1.0905s^{1.16}+1}e^{-0.783s}$ | 1.928 | 1.094 |

w/n:= without noise, n:= with noise

the collected step response. This routine aims to find the model parameters that would ideally reduce the integral of squared-time-weighted-error (ISTE) to zero. The MATLAB function `fsolve` is used to calculate the best estimated parameters which satisfies the above objective function in (28).

Following benchmark examples from literature are considered to show the effectiveness of Walsh functions for solving fractional order differential equations. Two examples are fractional-order systems and the other two are integer order systems.

$$g_1(s) = \frac{1}{s^{0.6}+2}e^{-0.4s} \tag{29}$$

$$g_2(s) = \frac{2s^{0.8}+1}{10s^{1.6}+2s^{0.8}+1}e^{-3.2s} \tag{30}$$

$$g_3(s) = \frac{4s+1}{(9s^2+3s+1)(s+1)}e^{-0.5s} \tag{31}$$

$$g_4(s) = \frac{2s+1}{(s+1)^3}e^{-0.5s} \tag{32}$$

### A. Numerical analysis

In order to assess the performance, both the time and frequency domain errors are calculated for all examples studied. In some cases, even though identification method give the close approximations in one domain but deviates too far away from the real system response in other domain. To represent better performance, the identification error should be small in both time and frequency domains. Moreover, the closeness of estimated model and true model can be demonstrated via step response in time domain and the Nyquist plot in frequency domain.

The fitting error of the identified models in comparison to techniques in recent literature [16], [15] is justified through the time domain identification error, $E_t$ and frequency domain identification error, $E_f$. The widely used fitting error of time domain response over the transient period,

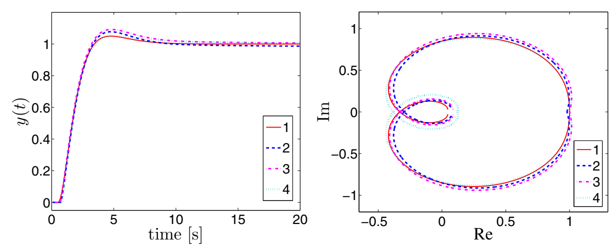
$$E_t = \frac{1}{M}\sum_{k=1}^{M}\left[y(k) - ydata(k)\right]^2, \tag{33}$$

and of the frequency response

$$E_f = \max_{\omega\in[0,\omega_c]}\left\{\left|\frac{\hat{g}(j\omega)-g(j\omega)}{g(j\omega)}\right| \times 100\%\right\} \tag{34}$$

can be calculated to quantitatively evaluate the accuracy. Here, $g(j\omega)$ is the frequency response of the actual system, $\hat{g}(j\omega)$ is the frequency response of the identified model and $\omega_c$ is the cutoff angular frequency when phase of $g(j\omega_c)$ equals to $-\pi$.

Table I shows the identified transfer function models with/without measurement noise along with respective models derived using other methods [16], [15]. For all examples, even a low order system model developed by the proposed method is very close to that of the actual system. Fig.1 shows step response and Nyquist plot comparison for $g_4(s)$. The proposed method resulted a much better match of step response in the time domain, also provided an excellent match in the frequency domain.




(a) (b)

Fig. 1. Nyquist plots for $g_4(j\omega)$: 1. actual 2. proposed 3. TPD [15] and 4. SVS [15]

### B. Effect of different types of input excitation

Generally, step signal is used as an input to generate output $ydata$. To test the proposed method, different input signals have been used. A pseudo-random binary sequence (PRBS) signal is used as an input signal for $g_1(s)$ as shown in Fig. 2. Again, the proposed method identifies the model very close to the actual system. Also, with mix sine wave input $u(t) = \sin 5t + 0.5\sin 3.1t + 0.6\sin t$, it estimates accurate model for $g_4(s)$ as shown in Fig.3.

### C. Effect of noise

Noise is an unavoidable factor of the real-world systems. Therefore, to get the sense of realistic condition, it is essential to validate the identification technique in the presence of noise to design robust control for the given system. To verify the usefulness of proposed method, the system model is estimated in the face of measurement

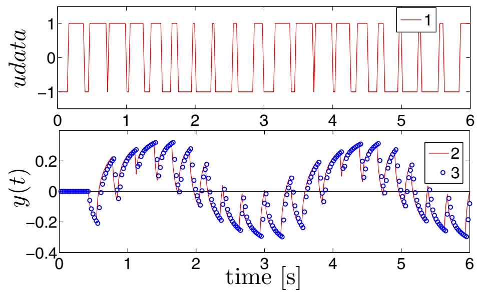



Fig. 2. $g_1(s)$: 1. PBRS input, 2. actual and 3. identified model

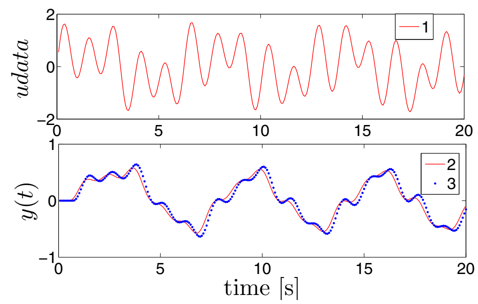



Fig. 3. $g_4(s)$: 1. sinusoidal input, 2. actual and 3. identified model

noise. Let the datasets be corrupted by Gaussian distributed random noise with signal-to-noise ratio (SNR) value of 20 dB. Fig. 4 compares the step responses of the noisy signal and fractional-order model obtained with the presented method for $g_2(s)$. As can be seen, the method gives a much better match of the actual system output even though the output data is noisy. Also, for $g_1(s)$, Fig.5

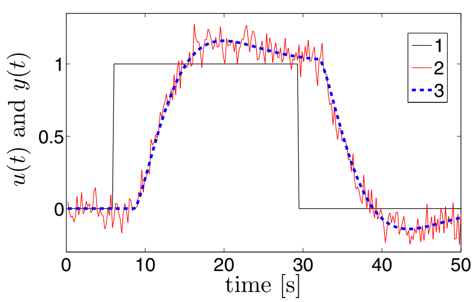



Fig. 4. $g_2(s)$: 1. input signal, 2. noisy output and 3. identified model

shows normalized mean values of all parameters and their possible range of deviation for different values of SNR. Normalized value of parameter is obtained by taking ratio of estimated to true parameter value. The approximation accuracy increases with the increase of SNR.

## D. *Effect of data length*

The effect of data length $M$ is verified for $g_1(s)$. Fig. 6 shows normalized mean values of all parameters and

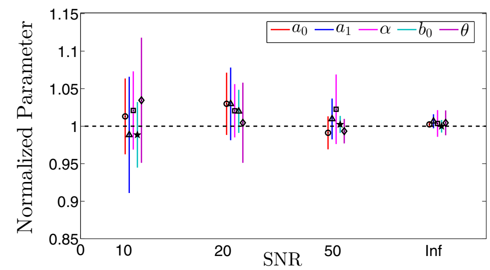



Fig. 5. Effect of different SNR on parameter identification

their possible range of deviation for different values of $M$. But, for the higher values of $M$, more time is required for identification. So, the selection of $M$ is a tradeoff between accuracy and time. However, for $M = 256$ acceptable values of parameters can be obtained with reasonable speed of the identification process.

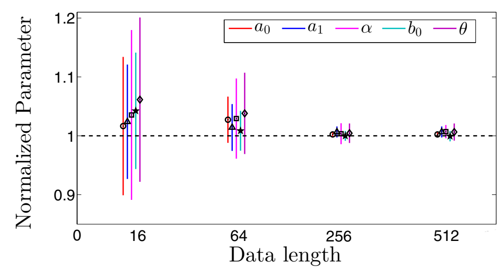



Fig. 6. Effect of data length on parameter identification

## E. *Real-time experiment*

To represent the applicability of the proposed method, experimental GSMT2014 DC servo system control platform [21] was considered. GSMT2014 is the system of twin motors based on high-performance motion controller GT-400 and the intelligent servo drive. The system is interfaced with computer for control and data access. The high-performance motion controller GT400 in the GSMT2014 enables the real control under MATLAB/Simulink. The laboratory setup for DC servo motor is depicted in Fig. 7. The rated voltage of servo motor is 26V and highest rotation speed is 3000 rpm. For this experiment, “speed” was selected as control parameter and experiment was performed in open-loop configuration. The step input excitation was used as an input with two different values and output speed data had been collected for identification. Using available data, the nominal fractional order transfer function of real system was identified as shown in Table II. One can compare time responses of identified fractional model with IO model in Fig. 8 and Table II. Again, results clearly reveal that the fractional model represents the actual system more closely than IO model.

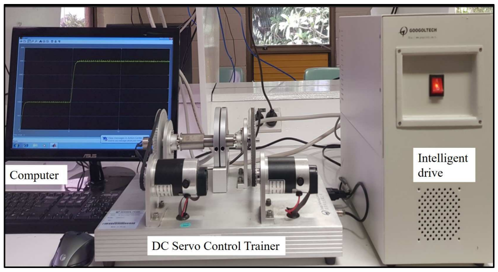



Fig. 7. Experimental setup for DC servo system control platform

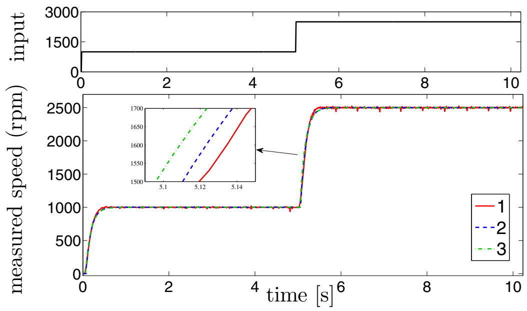



Fig. 8. Experimentation 1. Real data 2. Identified fractional order proposed model and 3. Identified integer order model [21]

TABLE II

DC SERVO SYSTEM: IDENTIFIED FO AND IO MODELS

| Model type | Identified model | $\varepsilon_t$ |
|---|---|---|
| FO | $G_{ff}(s) = \frac{0.999}{0.1199s^{1.010}+0.999}e^{-0.05198s}$ | $2.1288e+02$ |
| IO [21] | $G_{if}(s) = \frac{1}{0.12s+1}e^{-0.052s}$ | $4.5593e+02$ |

## VII. CONCLUSION

In conclusion, Walsh operational matrices can help to transform the fractional differential equations into algebraic equations and this facilitates the calculation. Another merit from the presented technique is to estimate all unknown model parameters including time delay simultaneously. But, its disadvantage is higher computational time for large value of $M$. From the results obtained and the discussions of previous section, it can be concluded that the proposed method identified the fractional-order time delay systems accurately without complex calculations of fractional derivative of input and output signals. The procedure can deal with the noisy data without additional filtering or signal smoothing. So, this technique can be used as a key component for the system modeling.

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