Modeling and parametric identification of Hammerstein systems with time delay and asymmetric dead-zones using fractional differential equations

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A B S T R A C T

The parametric identification of Hammerstein structured nonlinear systems with discontinuous asymmetric (two segment piecewise-linear with a dead-zone) nonlinearity and input time delay is presented using a fractional-order modeling technique. The effect of the unknown dead-zone nonlinearity is separated from the linear dynamics using special excitation signals to simplify the identification process. The use of fractional calculus permits reduced order modeling of the linear dynamics. An interactive block-based strategy is then introduced to simultaneously estimate the separated nonlinear and linear parameters, including the fractional differentiation order(s) and input time delay, which usually require separate algorithms in such cases. The proposed method utilizes block pulse functions to mitigate the computational complexity of fractional differential operations. Numerical examples are provided to validate the efficacy in comparison to existing methods. A DC servo motor application is used to demonstrate the proposed nonlinear modeling approach.

1. Introduction

The meticulous identification of complex dynamical systems for accurate modeling of the system characteristics is an intrinsic component of control design. With the exception of some atypical systems which exhibit linear characteristics due to a nominal operating point, majority of the existing industrial systems are inherently nonlinear in behavior. A nonlinear model definitely adds a certain level of complexity to the identification process compared to linear models. The development of identification strategies for nonlinear systems is therefore a crucial area of research. The Hammerstein model is a simple series nonlinear structure constituting of one input static nonlinear element succeeded by a dynamic LTI subsystem, and has been successfully utilized to adequately model various industrial systems [1–5].

The dead-zone is a common form of discontinuous actuator nonlinearity encountered in frequent industrial electromechanical components such as DC servo motors [6], hydraulic servo valves [7,8], ultrasonic motors [9], and in gear transmission servo systems [10]. The Hammerstein models of such systems with asymmetric dead-zone input nonlinearity have received much research attention due to it being an efficacious model for several electrical and mechanical systems previously stated. Initial work proposed in [11] described the dead-zone model solely for inputs that had an absolute value greater than the unknown dead-zone, resulting in redundant parameters. A new approach was then proposed in [12] using a decomposition technique. A deterministic approach based on separable least squares was proposed in [13]. A recursive least squares identification scheme for Hammerstein systems with dead-zone (and preload) was proposed in [14] yet the computation of the internal parameters was deemed complex and the proof of...
convergence was not provided due to the presence of these nonlinearity parameters. The recursive least squares technique was further improved in [15–17]. Non-iterative identification methods for Hammerstein models with input dead-zone were proposed in [18,19].

More recently, a gradient-based estimation algorithm [20], and a robust recursive algorithm using two improved recursive least squares algorithms [21] have been proposed. It is to be noted that all aforementioned methods were reported for the conventional Hammerstein model i.e. linear dynamics were of integer-order (IO) and characterized using ordinary differential equations.

In this paper, the Hammett model is extended to the fractional case, and a formulation of the fractional-order (FO) Hammerstein is deduced whereby the linear dynamic subsystem is characterized using fractional differential equations. Numerous modeling experiences in various fields of science and engineering often acquire a system model whose behavior is dissimilar to the one expected as IO models are used which have limited capability when modeling systems with memory phenomena. The orders of the system would naturally not be fixed integers but rather arbitrary real numbers and thus, incorporating fractional calculus would logically allow more accurate modeling of the system behavior. The FO Hammerstein model was initially explored in [22], and various contributions have subsequently followed [23–26]. Most recently, a novel identification strategy utilizing two filters for obtaining the derivative of the continuous signal was reported in [27]. However, majority of the fractional approaches stated do not consider a low-order model which negates the advantage. Furthermore, most approaches require the order to be commensurate which is not ideal and is practically infeasible. These limitations for FO Hammerstein model identification were overcome in [28] and [29] as low-order modeling techniques were proposed and the fractional operator represented with a generalized operational matrix through orthogonal basis functions, such as the block pulse (BPF) and Haar wavelets respectively. The fractional integral operation is converted into an algebraic expression which significantly reduces mathematical complexity. Moreover, a time-delay with single fractional pole model was utilized recently to accurately describe thermal process dynamics while considering the actuator rate limit [30].

Nevertheless, almost all contributions involving the fractional case of the Hammerstein model only consider the generalized case whereby the nonlinearity is represented with a continuous function. Identification strategies for Hammerstein systems with discontinuous nonlinearity using non-integer modeling techniques are yet to be reported. Therefore, a novel interactive block-based (IBB) identification strategy is proposed for FO models of Hammerstein systems with dead-zone nonlinearity utilizing block pulse operational matrices (BPOM). A single pole FO model with time delay is utilized for linear dynamics representation and the asymmetric dead-zone is interactively identified with all the linear dynamics parameters (coefficients, order and delay). This concurrent estimation of all parameters is permitted by block pulse representations of the known input and output signals, allowing complex mathematical operations to be written in simple algebraic form. The proposed IBB method utilizes a special excitation signal and accurately models the system with minimum number of parameters while imposing no prior knowledge requirements for identification. This FO modeling technique is compared with classical modeling techniques for the Hammerstein structure, and is experimentally validated on a practical DC servo system.

The paper is organized as follows: Section 2 provides some mathematical preliminaries. In Section 3, the problem description for the FO Hammerstein model with dead-zone input nonlinearity is introduced. Section 4 discusses the proposed identification scheme and Section 5 details the IBB identification strategy. The validation and performance analysis of the reported strategy is provided in Section 6. Finally, the paper concludes in Section 7.

2. Mathematical preliminaries

2.1. Fundamentals of FO systems

2.1.1. FO integrals

The FO integral with order α is a natural consequence of iterated integrals described as Cauchy’s formula, which simplifies n-fold integral of function x(t) to a convolution and is expressed as [31]

\[(\mathcal{I}_t^\alpha x)(t) = \left(\mathcal{I}_t^\alpha \mathcal{I}_t^{\alpha-n} x\right)(t) = \frac{1}{(n-1)!} \int_{t_0}^{t} (t - r)^{n-1} x(r) dr, t > t_0, n \in \mathbb{Z}^+.\] (1)

Considering the dynamic systems to be causal (t₀ = 0) and introducing the positive real number μ and gamma function \(\Gamma(n) = (n-1)!\), while extending to \(n \in \mathbb{R}^+\) results in the Riemann–Liouville (RL) definition for FO integral [31],

\[(\mathcal{I}_t^\alpha x)(t) = \frac{1}{\Gamma(n)} \int_0^t (t - r)^{n-1} x(r) dr\] (2)

for \(t > 0\) and \(\alpha \in \mathbb{R}^+\). The FO integral in (2) expressed as a causal convolution is

\[(\mathcal{I}_t^\alpha x)(t) = \frac{r_{t+}^{\alpha-1}}{\Gamma(\alpha)} * x(t), \quad \alpha \in \mathbb{R}^+,\] (3)

where \(r_{t+}^{\alpha-1} = 0\) for \(t < 0\), and \(r_{t+}^{\alpha-1} = t^{\alpha-1}\) for \(t \geq 0\).

2.1.2. FO derivatives

The FO derivative operator \(\mathcal{D}_t^\alpha\) can be expressed as the left inverse of the integral operator \(\mathcal{I}_t^\alpha\) through introduction of a positive integer r such that \(r - 1 < \alpha < r\). The RL definition of the FO derivative can then be given by [32]

\[(\mathcal{D}_t^\alpha x)(t) = \left(\mathcal{D}_t^\alpha \mathcal{I}_t^{-\alpha} x\right)(t) = \frac{d^r}{dt^r} \left(\frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{x(r)}{(t - \tau)^{\alpha+r-1}} d\tau\right),\] (4)

where \(\alpha \in \mathbb{R}^+\) and \(r \in \mathbb{N}\).
2.1.3. Laplace transforms

The use of the Laplace operator is fundamental in system modeling for control design. The RL definitions for the FO operators have been utilized in this paper. The Laplace transforms of the RL FO operators, with zero initial conditions, are

\[
\mathcal{L}[x(t)] = \frac{1}{s^\alpha} X(s), \quad \mathcal{L}[\mathcal{D}_t^\alpha x(t)] = s^\alpha X(s).
\]

(5)

2.2. BPOM of FO integration

The BPOM for fractional calculus was derived in [33] for the RL definition. The BPFs are a set of orthogonal functions \( \psi_\ell(t) \) with piece-wise constant values and contains \( F \) component functions. Considering a semi-open interval \([0, T)\) and \( T \) denoting matrix transpose, a set can be defined as [33]

\[
\psi_\ell(t) = [\varphi_0(t) \varphi_1(t) \cdots \varphi_{F-1}(t)]^T,
\]

where \( \varphi_j(t) \) is the \( j \)th component and is defined as

\[
\varphi_j(t) = \begin{cases} 1 & \text{if } \frac{jT}{F} \leq t < \frac{(j+1)T}{F} \\ 0 & \text{elsewhere}, \end{cases}
\]

where \( j = 0, 1, 2, \ldots, (F-1) \).

Any Lebesgue integrable time function \( x(t) \) can thus be expanded into a BPF series where \( t \in [0, T) \) with \( F \) terms. Considering the RL definition of the fractional integral in (1), if \( x(t) \) is expanded into BPF series, the fractional integral becomes

\[
(\mathcal{D}_t^\alpha x)(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * x(t) \approx W^T \frac{1}{\Gamma(\alpha)} [x^{\alpha-1} * \psi_\ell(t)],
\]

(8)

where \( W \) defines matrix

\[
W_j = \frac{T}{F} \int_{jT/F}^{(j+1)T/F} x(t)dt,
\]

and the BPOM of the fractional integrator can be derived [33] and defined as

\[
B_a = \left( \frac{T}{F} \right)^\alpha \frac{1}{\Gamma(\alpha + 2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{F-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{F-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \xi_1 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix},
\]

(11)

where \( \xi_i = (i + 1)^{\alpha+1} - 2i^{\alpha+1} + (i - 1)^{\alpha+1}, (i = 1, 2, \ldots, (F-1)) \). This can be considered to be the generalized BPOM of fractional integration as \( a = 1 \) would make \( B_a \) equal to the classical BPOM of integration. The generalized BPOM of fractional integration permits the fractional integral of any Lebesgue integrable time function \( x(t) \) to be written as

\[
(\mathcal{D}_t^\alpha x)(t) \approx X^T B_a \psi_\ell(t),
\]

(12)

where \( T \) denotes the transpose and \( X^T = [x_1, x_2, \ldots, x_F] \) is the coefficient vector.

2.3. BPOM for delay operator

In accordance with (8), the following delayed BPF can be obtained by expanding any absolutely integrable function containing a time delay \( x(t - \tau_L) \),

\[
x(t - \tau_L) \approx X^T \psi_\ell(t - \tau_L) \approx \sum_{i=1}^{F} x_i \psi_i(t - \tau_L),
\]

(13)

with the coefficient vector \( X^T = [x_1, x_2, \ldots, x_F] \) defined as

\[
x_i = \frac{F}{T} \int_0^T x(t - \tau_L) \psi_i(t - \tau_L) dt.
\]

(14)

The function \( \psi_\ell(t - \tau_L) \) is the shifted form of the BPF \( \psi_\ell(t) \) in (6) and can be written as

\[
\psi_\ell(t - \tau_L) = D \psi_\ell(t).
\]

(15)
Fig. 1. FO model of Hammerstein system with dead-zone.

Fig. 2. Piece-wise asymmetric dead-zone nonlinearity.

where $t > \tau_L$ and $0 \leq t \leq T$, and $D$ is the BPOM for the delay and is defined as [34]

\[
D = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_{F \times F}
\] (16)

The RL fractional integration of (15) using (2) results in

\[
(I_0^\alpha \psi_F)(t - \tau_L) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \psi_F(t - \tau_L).
\] (17)

Following some simplification using (10), (17) can be expressed in the matrix form

\[
(I_0^\alpha \psi_F)(t - \tau_L) = B_{\alpha} \psi_F(t - \tau_L),
\] (18)

with $B_{\alpha}$ defined in (11). Using (15), (18) can be written as

\[
(I_0^\alpha \psi_F)(t - \tau_L) = B_{\alpha} D \psi_F(t).
\] (19)

3. FO model of Hammerstein system

The FO model of Hammerstein systems with input dead-zone nonlinearity is shown in Fig. 1.

The conventional structure for a Hammerstein system is utilized where $u_t$ is the input excitation signal, $y_t$ is the system output and $v_t$ is the immeasurable internal signal i.e distortion of the input according to the dead-zone dynamics.

3.1. Parametric dead-zone model

The dead-zone can be represented as a static function $f(\cdot)$ that distorts the input signal i.e. it is the relationship between the actuator input $u_t$ and actuator output $v_t$. The actuator output will be zero (system is non-responsive) for a certain range of the input (the dead-zone), and for input values outside of this range, the output $v_t$ is a function of the input. The linear model (constant slope) is considered in this case and the analytical expression of the piecewise asymmetric dead-zone function is given by

\[
v_t = f(u_t) = \begin{cases}
m_r(u_t - d_r) & u_t > d_r \\
0 & d_l \leq u_t \leq d_r \\
m_l(u_t - d_l) & u_t < d_l,
\end{cases}
\] (20)

where $d_r \geq 0$ and $d_l \leq 0$ are the break-points, and $m_r, m_l > 0$ are the linear segment slopes. A graphical representation of the piece-wise asymmetric dead-zone function is displayed in Fig. 2.
It can be assumed without loss of generality that the point when the input is zero would occur inside the dead-zone \((d_i, d_f)\) as with a redefinition of input \(u\), this will always be attained. The parametric form of (20) can be written as [16]

\[
v_i = m_i \cdot u_i h(u_i) - m_i d_i \cdot h(u_i) + m_i u_i h(-u_i) - m_i d_i \cdot h(-u_i) - m_i (u_i - d_i) h(u_i) h(d_i - u_i) - m_i (u_i - d_i) h(-u_i) h(u_i - d_i),
\]

where \(h(\cdot)\) is a switching function and is defined for the input signal as

\[
h(u_i) = \begin{cases} 
1 & u_i > 0 \\
0 & u_i \leq 0.
\end{cases}
\]

This can equivalently be written in the parametric form (for \(u_i \neq 0\))

\[
h(u_i) = 0.5(1 + \text{sgn}(u_i)).
\]

For convenient parametrization, (21) can be rewritten to describe the intermediate variable as

\[
v_i = \theta_N^T \bar{A}_i - \gamma_i,
\]

where

\[
\theta_N = [\tau, m_i, m_i d_i, m_i, m_i d_i]^T,
\]

\[
\bar{A}_i = [u_i h(u_i), h(u_i), u_i h(-u_i), h(-u_i)],
\]

\[
\gamma_i = m_i (u_i - d_i) h(u_i) h(d_i - u_i) - m_i (u_i - d_i) h(-u_i) h(u_i - d_i).
\]

The vector \(\theta_N\) is the unknown dead-zone parameter vector, \(\bar{A}_i\) is the vector containing the input information and \(\gamma_i\) constitutes the remainder of distortion information.

### 3.2. FO linear model with time delay

Contrary to previous identification strategies for Hammerstein systems with dead-zone input nonlinearity, the linear subsystem is to be modeled with a FO and a time delay. Hence, the linear subsystem can be generally described by the following input–output relationship

\[
y_i = \frac{B(\varphi^l)}{A(\varphi^s)} \varphi^{s \tau_L - \tau_L}.
\]

where \(\tau_L\) represents the time delay, and \(B(\varphi^l)\) and \(A(\varphi^s)\) are respectively the input and output polynomials denoting the FO differential equation, and are expressed as [38]

\[
A(\varphi^s) = a_n \varphi^n_s + a_{n-1} \varphi^{n-1}_s + \ldots + a_1 \varphi^1_s + a_0
\]

\[
B(\varphi^l) = b_m \varphi^m_l + b_{m-1} \varphi^{m-1}_l + \ldots + b_1 \varphi^1_l + b_0
\]

where the orders are \(a_i, j = n, n - 1, \ldots, 1 \in \mathbb{R}^+, b_q, q = m, m - 1, \ldots, 1 \in \mathbb{R}^+, \) and the coefficients \(a_i (i = n, n - 1, \ldots, 0), b_q (q = m, m - 1, \ldots, 0)\) are real constants, and \(a_0 > b_m\).

For practicality, the Hammerstein system is modeled as a black-box i.e no commensurate information about the system dynamics is available and the identification is to be solely performed using known system input and output data. Hence, it is imperative to describe the system dynamics with adequate models. Since the linear subsystem is to be modeled with a FO, a model with low number of poles is highly desirable to avoid the memory constraints of FO implementation in practical applications. Low-order models are common for control design utilization in industries, and high-order dynamics are usually reduced to low-order for increased stability and compactability. This simultaneously reduces computational complexity and simplifies the control design process. Furthermore, for the Hammerstein model shown in Fig. 1, the parameters of the nonlinear and linear dynamic subsystems cannot be uniquely identified, as identical input and output responses would be produced for the pair \([\kappa n(u_i), G_{\kappa n}^s]\), where \(\kappa\) is any nonzero finite constant. In order to get a unique parametrization without loss of generality, the transfer function is normalized by assuming \(a_0 = 1\) [36], concurrently reducing the dimensionality of the optimization problem. Therefore, in this paper, the generalized FO linear model in (26) is reduced to a single pole form \((n = 1, m = 0)\) with the following continuous transfer function,

\[
G(s^s) = \frac{Y(s)}{V(s)} = \frac{b_0}{a_1 s^i + 1} e^{-\tau_L s}
\]

The unknown linear parameter vector can now be defined as \(\theta_L = [b_0, a_1, \tau_L]\).

### 3.3. Issue of dead-time merging with time delay

The time delay of FO linear dynamics can be realized from the BPOM in (16). However, in this case, the delayed input of the linear subsystem is the distorted system input, and the dead-zone introduces a dead-time \(\tau_D\). Therefore, the delay observed in the output of the linear subsystem would actually be an combination of the dead-time and the time delay \(\tau_L\). This effect is graphically displayed in Fig. 3. The combined delay can be represented by \(\tau\) where \(\tau = \tau_L + \tau_D\). Considering that the system is modeled...
Fig. 3. Effect of dead-zone with time delay in Hammerstein system.

Fig. 4. Input test signals.

as a black-box and no prior information about the dead-time or the time delay is known, the identification of the delay parameter becomes complex, due to the overall system output delay being a fusion of the dead-time and linear subsystem input delay.

It is assumed that due to this added complexity, majority of the proposed techniques for identification of Hammerstein systems containing dead-zone in the cited literature do not consider modeling the linear subsystem to have a time delay. However, one such system was considered with delay in [14], whereby the Hammerstein model was decomposed using the key separation principle, separating the parameters of the nonlinear dead-zone function and the linear dynamic subsystem. The separation of the nonlinear and linear parameters therefore mitigates the aforementioned merging issue. In this paper, the separation of the parameters is performed via a special test signal.

4. Proposed identification scheme

A new identification strategy based on separating the identification of the dead-zone parameters from that of the fractional linear subsystem is developed. This permits freedom to develop identification techniques for the specific blocks while avoiding over-parametrization, a common issue when utilizing the input–output relationship of the whole system for identification.

4.1. System activation

For separation of the nonlinearity effect from the linear subsystem, a special test signal composed of a binary signal and multi-sine signal is developed to activate the nonlinear system. The test signal is shown in Fig. 4 as an example, whereby the input signal $u_t$ constitutes of a two-step (binary) signal from $t_0 < t < t_1$ seconds with one step value being zero and the other step being nonzero, and a multi-sine signal from $t_1 < t < t_2$ seconds, where $t_0 \geq 0$ and $t_0 < t_1 < t_2$.

The input is separable, and defining the binary signal input as $u_1$, with $u_2$ as the multi-sine signal input, the corresponding system outputs can be defined as $y_1$ and $y_2$, respectively. The excitation periods of $u_1$ and $u_2$ are dependent on the number of component functions $F$ used to expand the signals into a BPF series. Therefore, this value of $F$ would correspond to the data-length of the two signal vectors. The system can therefore be excited separately whereby the data-length of each signal would be $F$, and thus the
excitation periods of $u_1$ and $u_2$ would be given by $FT_z$, where $T_z$ the sampling interval in seconds. In Fig. 4, the combined signal is presented whereby the overall data-length is $2F$. The excitation period of this combined signal would thus be equal to $2FT_z$ seconds.

The binary signal activates the FO linear model without activating the nonlinear function, since it can be assumed (without loss of generality) that the input to the linear subsystem would be the same as the system input i.e. $v_1 = u_1$. This is possible since the coefficients of the static polynomial nonlinear function can be multiplied with a nonzero constant $\lambda$, the reciprocal $1/\lambda$ can be multiplied to $B$ in (27) which is just 1 after normalization (28). This ensures that the input–output mapping remains unaltered [37]. This same phenomena can also alternatively be achieved by using a step signal as $u_1$. The multi-sine input is used to persistently excite the Hammerstein system, thus activating the nonlinear distortion. It should be noted that any nonlinear signal could be used for $u_2$, provided the amplitude is such to activate the dead-zone (Assumption 4).

However, the following assumptions need to be outlined when utilizing this proposed test signal:

**Assumption 1.** The input signal $u_1$ and output signal $y_1$ are absolutely integrable functions which satisfy the conditions $u_1 \in L^2(0,T)$ and $y_1 \in L^2(0,T)$, where $L$ is the Lebesgue integral and $T$ is the total time period.

**Assumption 2.** The system is stable with bounded system input $u_i$ and bounded output $y_i$.

**Assumption 3.** The binary signal $u_1$ and the multi-sine signal $u_2$ persistently excite the FO linear model and the overall Hammerstein system respectively. Therefore the input signal $u_1$ is persistently exciting i.e there exists a constant $\kappa_0$ and positive constants $T$ and $\mu$, such that for all $\kappa \geq \kappa_0$,

$$\frac{1}{T} \int_{\kappa}^{\kappa+T} u_1^2 dt \geq \mu I > 0$$

(29)

**Assumption 4.** The amplitude of the input signal $u_2$ is such that the dead-zone function is fully activated i.e $u_{2_{\max}} > d_f$ and $u_{2_{\min}} < d_f$

**Remark 1.** Assumption 1 allows the input and output signals to be transformed into an algebraic expression using the generalized BPOM.

**Remark 2.** Assumption 2 confirms the existence of the system as well as safety when measuring the data.

**Remark 3.** Assumption 3 ensures the convergence of parameters during identification [38].

**Remark 4.** Assumption 4 guarantees that the system output will contain all the effects of the distortion, ensuring accurate identification of the dead-zone. The input amplitude in physical systems can be adjusted to obtain the expected system response.

The identification strategy is divided into two stages; identification of the linearized model (separated time delay parameter), followed by using the estimated linear parameters to interactively identify the nonlinear parameters while tuning the linear parameters. It is to be noted that when considering the linear dynamics without time delay, the first stage is not necessary and the whole nonlinear system can identified by just exercising the second stage.

4.2. Identification of the linearized model

The output $y_1$ of the linearized model is obtained through excitation of the system with $u_1$. The nonzero constant $\lambda$, in this case, would be the resulting nonlinear gain that is multiplied to the linear subsystem i.e the output of the dead-zone function when the input is a two step binary input $u_1$, and this gain would be $\lambda = m_i(a_{i_{\max}} - a_{i_{\min}})$. Therefore $v_1 = \lambda u_1$, and if $v_1 = u_1$ then (28) becomes,

$$G(s) = \frac{Y_1(s)}{U_1(s)} = \frac{\eta}{a_i s^{a_i} + 1} e^{-\tau_{s} s}$$

(30)

where $\eta = \lambda b_0$.

To begin with, the input to the linear subsystem and output are expanded into BPF using (13) and (14) as

$$u_1(t - \tau_L) \cong \sum_{j=1}^{F} a_j \psi_j(t) = U_1^T \psi_1 F(t - \tau_L)$$

(31)

$$y_1(t) \cong \sum_{j=1}^{F} v_1 \psi_j(t) = Y_1^T \psi_1 F(t)$$

(32)

where $U_1^T = [u_1, u_1, \ldots, u_1]$ and $Y_1^T = [y_1, y_1, \ldots, y_1]$. The R-L integrals of (31) and (32) are

$$[\mathcal{F}_0^a u_1](t - \tau) \cong U_1^T \mathcal{F}_a \mathcal{F}_a \psi_1 F(t)$$

(33)

$$[\mathcal{F}_0^a y_1](t) \cong Y_1^T \mathcal{F}_a \mathcal{F}_a \psi_1 F(t)$$

(34)
Rewriting (31) in the fractional differential form as,
\[ y_1(t)(a_1 D^{a_1} + 1) = \eta u_1(t - \tau_L) \] (35)
and integrating both sides and using (33) and (34) results in
\[ Y_1^T(a_1 I + B_{\eta_1})\psi_f(t) = U_1^T \eta B_{\alpha_1} D \psi_f(t) \] (36)

Simplifying the aforementioned expression, the linearized output vector can be expressed as
\[ Y_1^T = U_1^T \eta B_{\alpha_1} D(a_1 I + B_{\eta_1})^{-1}, \] (37)
and substituting into (32) provides the algebraic expression of the output of the linearized model
\[ y_1(t) = U_1^T \eta B_{\alpha_1} D(a_1 I + B_{\eta_1})^{-1} \psi_f(t). \] (38)

Now that BPOM representation of the linearized output is available, the optimal parameters of the linear subsystem can be obtained by minimizing a suitable objective function. The estimate of the linearized output can be written as
\[ \hat{y}_1(t) = U_1^T \eta B_{\alpha_1} \hat{D}(a_1 I + B_{\eta_1})^{-1} \psi_f(t) \] (39)

Defining the first stage linear subsystem parameter vector \( \theta_{L_1} = [\eta, a_1, \alpha_1, \tau_L] \), the estimate of the linear subsystem parameters \( \hat{\theta}_{L_1} \) can then be obtained using the objective function,
\[ \text{ITSE}_{L_1} = \min_{\theta_{L_1}} \sum_{k=1}^{F} [k(y_1(k) - \hat{y}_1(k))]^2 \] (40)

where \( \text{ITSE}_{L_1} \) is the integral of time-squared error and \( y_1(k) \) is the response at time \( t = t_k \) with \( F \) data points. The function \texttt{fsolve} from the MATLAB® Optimization Toolbox is utilized to solve the optimization problem of (40) and obtain \( \hat{\theta}_{L_1} \) (consult Remark 8 for details).

**Remark 5.** An estimate of the time delay \( \tau_L \) is obtained and can be distinguished from the dead-time \( \tau_D \) from the overall delay of the nonlinear system.

**Remark 6.** The estimate of \( b_0 \) is yet to be obtained, however, the estimate of \( \eta \) contains information regarding the value of \( b_0 \). The gain \( \lambda \) would be estimated in the following steps and subsequently, the estimate of \( b_0 \) would be obtained.

### 4.3. Identification of the nonlinear model

An IBB identification strategy is proposed to separately but simultaneously estimate the dead-zone and FO linear model parameters. The key principle of the strategy is to consider the representation of the immeasurable intermediate signal \( \psi \) in two forms; firstly as the delayed input to the linear subsystem i.e. with respect to the measured system output, and secondly, as the output of the dead-zone nonlinear function i.e. with respect to the system input. This is rendered possible due to the representation of the signals using BPF and successively utilizing the signals in their algebraic form, allowing the complex nature of the identification problem to be simplified into simple algebraic operations.

The idea is to identify the dead-zone block parameters with \( \psi \) replaced with the estimate computed from the linear block, and when identifying the linear block, \( \psi \) is replaced with the estimate computed from the dead-zone block. For this stage of the identification, the input–output data-sets \( (u_2, y_2) \) are utilized. The output data \( y_2 \) contains the overall nonlinear system response information and thus can be used to estimate the characteristics of the intermediate variable i.e. \( \psi \), in this case.

#### 4.3.1. The \( v \) – \( u \) nonlinear relationship

Examining the nonlinear relationship, the intermediate signal \( v_2 \) can be represented in terms of the dead-zone parameters i.e. as the output of the nonlinear block, and is denoted as \( v_{2_N} \). Expanding the input and output signals of the dead-zone block into BPF yields
\[ u_2(t) \equiv \sum_{j=1}^{F} u_{2_j} \psi_f(t) = U_2^T \psi_f(t) \] (41)
\[ v_{2_N}(t) \equiv \sum_{j=1}^{F} v_{2_N,j} \psi_f(t) = V_2^T \psi_f(t) \] (42)
where \( U_2 = [u_{2_1}, u_{2_2}, \ldots, u_{2_F}] \) and \( V_2 = [v_{2_N,1}, v_{2_N,2}, \ldots, v_{2_N,F}] \). Therefore, using (21), the intermediate signal vector is given by
\[ V_{2_N}^T = m_r \cdot U_2^T H_u + m_r \cdot d_r \cdot H_u + m_r \cdot U_2^T H_u - m_r \cdot d_r \cdot H_u \]
\[ -m_r \cdot (U_2^T - d_r) H_u H_d_r - m_l (U_2^T - d_l) H_u H_d_l \] (43)
where \( \mathbf{H}_d = h[\mathbf{U}_d^T], \mathbf{H}_w = h[\mathbf{U}_w^T], \mathbf{H}_d = h[d_d - \mathbf{U}_d^T] \) and \( \mathbf{H}_{d_d} = h[\mathbf{U}_{d_d}^T - d_d] \). Substituting the aforementioned vector into (42) and representing in the form (24) gives intermediate variable as the algebraic expression

\[
v_{2,v}(t) = (\theta^T \mathbf{A} - \gamma)\psi(t) \tag{44}
\]

where

\[
\mathbf{A} = [\mathbf{U}^T \mathbf{H}_w, \mathbf{U}^T \mathbf{H}_d, \mathbf{U}^T \mathbf{H}_d^T], \quad \gamma = m_1(\mathbf{U}_d^T - d_d)\mathbf{H}_{d_d} - m_2(\mathbf{U}_d^T - d_d)\mathbf{H}_{d_d}. \tag{45}
\]

Considering \( \hat{\theta}_N \) to be the estimate of the dead-zone parameter vector and concurrently the vector \( \tilde{\gamma} \) to be the estimate of the remaining distortion information, the estimate of the immeasurable intermediate signal can be written as

\[
\hat{v}_{2,v}(t) = (\hat{\theta}^T \mathbf{A} - \tilde{\gamma})\psi(t) \tag{46}
\]

and a similar objective function such as (40) can be minimized to obtain the optimal dead-zone estimates. However, this is not currently possible as the intermediate signal must be known in order to minimize the error. Hence, the proposed solution is to replace the unknown \( v_2 \) signal with its estimate that is computed from the \( y - v \) linear relationship.

### 4.3.2. The \( y - v \) linear relationship

Examining the linear relationship, the intermediate signal \( v_2 \) can be represented in terms of the linear subsystem parameters i.e. as the input of the linear block. Re-writing (28) in terms of the data-set \( (u_2, y_2) \) yields

\[
G_{yv}(s) \equiv \frac{y_2(s)}{v_2(s)} = \frac{-b_0}{a_0s^{\alpha_1} + 1} e^{-\tau s}. \tag{47}
\]

Denoting the intermediate signal now as \( v_{2,L} \), the input to the linear subsystem and output are expanded into BPF using (6) and (13) as

\[
v_{2,l}(t) = \sum_{j=1}^{F} v_{2,l,j}\psi_j(t) = \mathbf{V}_{2,l}^T \psi(t)
\]

\[
y_{2}(t) = \sum_{j=1}^{F} y_{2,j}\psi_j(t) = \mathbf{Y}_{2,l}^T \psi(t)
\]

where \( \mathbf{V}_{2,l} = [v_{2,l,1}^T, v_{2,l,2}^T, \ldots, v_{2,l,F}^T] \) and \( \mathbf{Y}_{2,l} = [y_{2,1}, y_{2,2}, \ldots, y_{2,F}] \). The R-L integrals of (48) and (49) are

\[
[\int_0^T v_{2,l}(t) \, dt] \approx \mathbf{V}_{2,l}^T \mathbf{B}_L \mathbf{D} \psi(t)
\]

\[
[\int_0^T y_{2}(t) \, dt] \approx \mathbf{Y}_{2,l}^T \mathbf{B}_L \mathbf{D} \psi(t)
\]

Rewriting (50) in the fractional differential forms as,

\[
y_{2}(t)(t_i, \mathcal{G}^\alpha) = b_0 v_{2,l}(t - t_i)
\]

and integrating both sides and using (52) results in

\[
\mathbf{Y}_{2,l}^T (a_i \mathbf{I} + \mathbf{B}_{a_i}) \psi(t) = \mathbf{V}_{2,l}^T b_0 \mathbf{B}_{a_i} \mathbf{D} \psi(t).
\]

The aforementioned expression can thus be used to express the intermediate signal as a vector

\[
\mathbf{V}_{2,l}^T = \mathbf{Y}_{2,l}^T (a_i \mathbf{I} + \mathbf{B}_{a_i})^{-1} \mathbf{D}^T.
\]

It should be noted that since the last row of the operational matrix \( \mathbf{D} \) comprises of zeros as seen in (16), suggesting that the inverse of the matrix cannot be directly computed. However, the matrix is orthogonal, thus it is necessarily invertible i.e. the transpose is equal to the inverse.

The advantage of converting the fractional differential problem to algebraic operations for nonlinear system identification is now evident as the immeasurable intermediate has been expressed in terms of a matrix which can be estimated. Re-writing (50) as an algebraic operation using (32) yields

\[
v_{2,v}(t) = \mathbf{Y}_{2,l}^T (a_i \mathbf{I} + \mathbf{B}_{a_i})^{-1} \mathbf{D}^T \psi(t),
\]

and the estimate of the intermediate signal can be expressed as

\[
\hat{v}_{2,v}(t) = \mathbf{Y}_{2,l}^T (a_i \mathbf{I} + \mathbf{B}_{a_i})^{-1} \mathbf{D}^T \psi(t).
\]

The output vector of the nonlinearity activated system can be expressed as a vector by simplifying (53) and using vector form of (44) to replace \( \mathbf{V}_{2,l}^T \),

\[
\mathbf{Y}_{2,l}^T = (\theta^T \mathbf{A} - \gamma) b_0 \mathbf{B}_{a_i} (a_i \mathbf{I} + \mathbf{B}_{a_i})^{-1}.
\]
Defining the second stage linear parameter vector now as \( \theta_{L_2} = [b_0, a_1, r_L] \), it can be seen that (57) is the output representation is in terms of the dead-zone parameter vector and the linear subsystem parameter vector i.e. \( Y_z^T(\theta_N, \theta_{L_2}) \).

The algebraic expression of the system output using (51) is
\[
y_2(t) = (\hat{\Theta}_N^T A - \hat{\gamma})b_0B_{a_1}D(a_1I + B_{a_1})^{-1}\psi_F(t),
\]
and the estimate of the system output thus is given by
\[
\hat{y}_2(t) = (\hat{\Theta}_N^T A - \hat{\gamma})b_0B_{a_1}D(\hat{a}_1I + B_{a_1})^{-1}\psi_F(t).
\]

The estimates of the output \( \hat{y}_2 \) and intermediate variables \( \hat{v}_2^N, \hat{v}_2^L \) are utilized in an interactive manner to get the optimal estimates for dead-zone and linear subsystem parameters. The interactive nature of the estimation is fundamental to the overall identification scheme.

5. The interactive block-based strategy

Since an optimal estimate of the linear subsystem parameters \( \hat{\theta}_{L_1} \) was obtained by minimizing the objective function in (40), replacing the estimates in (56) with \( \hat{\theta}_{L_1} \) provides an initial estimate of the intermediate signal, \( \hat{v}_{2_1} \). Now the estimated intermediate signal considered from the linear relationship can be used to minimize an objective function. The objective function that can be used to estimate the optimal parameters can be written as,
\[
\text{ITSE}_N = \min_{\theta_N} \sum_{k=1}^{F} [k(\hat{v}_{2_N}(k) - \hat{v}_{2_N}(k))]^2.
\]
An initial estimate of the dead-zone would be required to initialize the optimization, and hence a random parameter vector \( \theta_{N_0} \) is utilized. Due to \( \hat{\theta}_{L_1} \) containing the optimal estimates of linearized model, it has been used to compute and minimize the error to obtain \( \theta_N \). Now this estimate of the dead-zone parameters can be used to replace \( \theta_N \) in the system output expression (59)(with linear parameters replaced with \( \hat{\theta}_{L_1} \), and the linear parameters can then be re-estimated (tuned) according to the actual nonlinear system response by minimizing the objective function similar to (40),
\[
\text{ITSE}_{L_2} = \min_{\theta_{L_2}} \sum_{k=1}^{F} [k(y_2(k) - \hat{y}_2(k))]^2.
\]

These estimated linear subsystem parameters \( \hat{\theta}_{L_2} \) can now be used to re-compute the estimate of the intermediate signal \( \hat{v}_{2_2} \) in (56) which can then be subsequently used to re-estimate the dead-zone in (60), with the previous \( \hat{\theta}_{L_1} \) being used to replace \( \theta_{N_0} \). The new estimates of the dead-zone are then used again in (59) and relatively in (61) to compute the new linear parameters estimates \( \hat{\theta}_{L_2} \), and the loop continues until the stopping criteria of (61) is met i.e. the error is minimum. The two optimizations algorithms (60) and (61) are run in a one recursive step to simultaneously provide the optimal dead-zone and linear subsystem parameters. The major steps for the complete identification system are summarized in Table 1.

Remark 7. The estimate of \( \eta \) undergoes tuning during optimization to obtain \( b_0 \) when the linear parameters are re-estimated according to the nonlinear system output data and the computed dead-zone parameters. The computed dead-zone parameters continually updates the nonlinear gain \( \lambda \) and the value of \( \eta \) is multiplied by the changing factor \( 1/\lambda \) as the optimization fits the estimated output response to the actual response, culminating in the final optimized value of \( \eta \) being the estimate of \( b_0 \). This ensures that the optimal estimate of \( b_0 \) is free of the nonlinear gain.

Remark 8. To solve the optimization problem in MATLAB®, the function `fsolve` is utilized, which is a nonlinear system solver. The Levenberg-Marquardt algorithm option was set for the solver in order to ensure convergence. However, appropriate initialization would improve the computational speed and accuracy of the optimization. There is currently no mathematical theory available for the appropriate selection of initial parameters. To initialize the optimization, random parameter vectors are chosen for \( \theta_{L_0} \) and \( \theta_{N_0} \). Subsequently, the obtained estimates after the first trial are then utilized to initialize the optimization for the succeeding trial.
and so on. Due to the convex nature of the mathematical optimization problems (40), (60) and (61), the parameter estimations will always converge to an optimal value in a finite number of trials. It should be noted that the selection of initial guesses affects the convergence rate and number of trials required for the optimization procedure. For the validation examples, $\theta_{L0} = [1, 1, 1]$ and $\theta_{N0} = [1, 1, 1, 1]$ are utilized for initialization.

**Remark 9.** The parameter vector for the dead-zone, unlike the linear parameter vector, is not comprised of the individual parameters. As seen in (25), the dead-zone parameters $d_r$ and $d_l$ are not individually present in the dead-zone parameter vector, rather a product of the gradient and the dead-zone is i.e $m_r d_r$ and $m_l d_l$. Therefore, an additional step of dividing the second and forth identified parameters respectively by the first and third is performed within the algorithm to get the individual dead-zone parameters.

**Remark 10.** The strategy proposed fixes the linear subsystem to a reduced single pole FO model with time delay. This is done for practical feasibility and reduction of the number of parameters to be estimated. However, the proposed scheme is flexible enough to cater to for higher order fractional models if desired ($n > 1, m > 0$). In this case, the input and output polynomials in (27) converted into an algebraic form using BPF, and normalized ($a_0 = 1$), would be written as

$$A(\cdot) = a_0 I + a_{n-1} B_{x_n - a_{n-1}} + \cdots + a_0 B_{x_n - a_0}$$
$$B(\cdot) = b_m B_{u_n - \beta_m} + b_{m-1} B_{x_n - \beta_{m-1}} + \cdots + b_0 B_{u_n - \beta_0}$$

and subsequently utilized to obtain the estimate of the signals in the form of algebraic operations. However, it should be noted that increasing the number of parameters to be estimated will increase the computation time and will affect the overall accuracy of the identification.

The complete proposed IBB identification scheme for FO Hammerstein models with input dead-zone nonlinearity is simplified for comprehension in Fig. 5. (Note: LSO — Linear Subsystem Optimization)

### 6. Validation

This section validates the performance of the proposed IBB identification scheme via a numerical example, a comparative study and a practical DC servo motor case.
of ITSE noise, the aforementioned system was identified with the proposed technique. The noisy cases were simulated for by noise. Considering an output-error model of the Hammerstein model shown in Fig. 1 with zero mean white Gaussian measurement strategy as the identification technique must be robust enough to accurately identify parameters with measurement data distorted and this is evident in Fig. 9. However, due to the difference being very minimal in nature, it is permissible to utilize value slightly faster than the latter for the initial optimizations (Trial 1 for stage one and Loop 1 (nonlinear function) for stage two)

where

\[
\begin{align*}
\Delta_t &= \frac{2}{3} \delta^{0.75} + \frac{1}{4} e^{-0.4t} \\
G(s) &= \frac{2}{3} \delta^{0.75} + 1 \\
\end{align*}
\]

where \( m = 3, d = 1.5, m_1 = 2 \) and \( d_1 = -1 \), and the FO linear model is

\[
G(s) = \frac{2}{3} \delta^{0.75} + 1
\]

where \( b_0 = 2, a_1 = 3, a = 0.75 \) and \( \tau_L = 0.4 \). The system was excited with the test inputs shown in Fig. 3 with \( u_{\text{max}} = 3 \) and \( (u_{\text{min}}, u_{\text{max}}) = (-3, 3) \). The data-length \( F \) was set to 512 samples.

Noise is inherent in almost all industrial systems and is a fundamental factor in determining the efficacy of an identification strategy as the identification technique must be robust enough to accurately identify parameters with measurement data distorted by noise. Considering an output-error model of the Hammerstein model shown in Fig. 1 with zero mean white Gaussian measurement noise, the aforementioned system was identified with the proposed technique. The noisy cases were simulated for SNR = 20 dB and SNR = 10 dB and compared with a noise free (SNR = \( \infty \)) case. The block pulse coefficients of the output signal obtained from measured and simulated data are evaluated and used in optimization problems (40) and (61) to assess the quality of the parameter estimations. Through minimization of the estimation error, improved estimations of the parameters are obtained and these values are used to simulate the estimated output. Due to this optimization process utilized in the presented IBB technique with alternating simulation and optimization, there is a decrease in the influence of noise. This robustness is inherent in the optimization process since the simulated output is forced to correspond to the measured output. Moreover, the use of BPOM avoids computation of complex fractional derivatives of the input and output signals. This explicit feature of the proposed method also contributes to the reduced noise sensitivity during identification. The identification results are tabulated in Table 2 and identification accuracy is evaluated using mean-squared error (MSE).

Fig. 6 compares the dead-zone functions and Fig. 7 displays the system response.

The proposed technique estimates the overall Hammerstein system utilizing just the measurement data. The identification errors are small even in the presence of noise, highlighting that the IBB technique is robust to measurement noise, and preprocessing of data is not essential to get a good estimation.

6.1. Example 1: Numerical study

This study is to confirm the functionality of the algorithm, the convergence of the parameters during identification, and performance of the algorithm with noisy data. Consider the following Hammerstein system, whereby the dead-zone nonlinearity is represented as

\[
e_i = \begin{cases} 
3(u_i - 1.5) & u_i > 1.5 \\
0 & -1 \leq u_i \leq 1.5 \\
2(u_i + 1) & u_i < -1.
\end{cases}
\]

where \( m_r = 3, d_r = 1.5, m_l = 2 \) and \( d_l = -1 \), and the FO linear model is

\[
\begin{align*}
\hat{a}_1 &= \frac{2}{3} \delta^{0.75} + 1 \\
\end{align*}
\]

where \( b_0 = 2, a_1 = 3, a = 0.75 \) and \( \tau_L = 0.4 \). The system was excited with the test inputs shown in Fig. 3 with \( u_{\text{max}} = 3 \) and \( (u_{\text{min}}, u_{\text{max}}) = (-3, 3) \). The data-length \( F \) was set to 512 samples.

The system was identified with the proposed technique. The noisy cases were simulated for SNR = 20 dB and SNR = 10 dB and compared with a noise free (SNR = \( \infty \)) case. The block pulse coefficients of the output signal obtained from measured and simulated data are evaluated and used in optimization problems (40) and (61) to assess the quality of the parameter estimations. Through minimization of the estimation error, improved estimations of the parameters are obtained and these values are used to simulate the estimated output. Due to this optimization process utilized in the presented IBB technique with alternating simulation and optimization, there is a decrease in the influence of noise. This robustness is inherent in the optimization process since the simulated output is forced to correspond to the measured output. Moreover, the use of BPOM avoids computation of complex fractional derivatives of the input and output signals. This explicit feature of the proposed method also contributes to the reduced noise sensitivity during identification. The identification results are tabulated in Table 2 and identification accuracy is evaluated using mean-squared error (MSE).

Table 2

<table>
<thead>
<tr>
<th>Model estimates</th>
<th>SNR</th>
<th>( \infty )</th>
<th>20 dB</th>
<th>10 dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{a}_1 )</td>
<td>2.9578</td>
<td>2.8400</td>
<td>2.7560</td>
<td></td>
</tr>
<tr>
<td>( \hat{a}_2 )</td>
<td>1.9654</td>
<td>1.8754</td>
<td>1.8541</td>
<td></td>
</tr>
<tr>
<td>( \hat{b}_1 )</td>
<td>1.5209</td>
<td>1.4589</td>
<td>1.4357</td>
<td></td>
</tr>
<tr>
<td>( \hat{d}_1 )</td>
<td>-0.9985</td>
<td>-0.9976</td>
<td>-0.9845</td>
<td></td>
</tr>
<tr>
<td>( \hat{b}_2 )</td>
<td>2.0210</td>
<td>2.0810</td>
<td>2.1650</td>
<td></td>
</tr>
<tr>
<td>( \hat{a}_1 )</td>
<td>2.9854</td>
<td>2.8400</td>
<td>2.8610</td>
<td></td>
</tr>
<tr>
<td>( \hat{a}_2 )</td>
<td>0.7451</td>
<td>0.7475</td>
<td>0.7397</td>
<td></td>
</tr>
<tr>
<td>( \hat{d}_1 )</td>
<td>-0.4004</td>
<td>-0.4011</td>
<td>-0.3895</td>
<td></td>
</tr>
<tr>
<td>MSE(×10^{-5})</td>
<td>0.0413</td>
<td>4.1147</td>
<td>23.1891</td>
<td></td>
</tr>
</tbody>
</table>

6.1.1. Convergence analysis

The convergence of the first stage optimization is shown in Fig. 8a. It proves the convergence of objective function ITSE\(_{L_1}\) towards a minimum possible value is achieved in 200 iterations over a finite number of trials. The initial optimization procedure (Trial 1) was performed and result was used as the initial guess for the succeeding trial and so forth. The convergence rate increased with each trial and optimum linear parameters were obtained upon termination (criterion satisfied). Fig. 8b displays the convergence of the second stage optimization, where the linear and nonlinear parameters are simultaneously estimated. Both the objective functions ITSE\(_N\) and ITSE\(_L_1\) converge towards a minimum possible value in 200 iterations over a finite number of loops. The termination criterion was met after three consecutive loops i.e. the loop is exited when the error of estimated overall response (ITSE\(_L_1\)) was less than \( 1 \times 10^{-6} \). The ITSE of the estimated intermediate signal significantly decreased after the initial loop and converged to a minimum value in parallel to the estimated overall response, thus providing optimum estimates of the dead-zone parameters. The ITSE was preferred over the integral squared error (ISE) for the objective functions as it was observed that the former converged to a minimum value slightly faster than the latter for the initial optimizations (Trial 1 for stage one and Loop 1 (nonlinear function) for stage two) and this is evident in Fig. 9. However, due to the difference being very minimal in nature, it is permissible to utilize ISE in place of ITSE for the proposed IBB technique.
6.1.2. Effect of data-length on accuracy

The accuracy of the IBB identification technique is also significantly dependent on the approximation of the BPOM, which is relative to the data-length $F$. The effect of various data-lengths on the accuracy is displayed in Table 3. It can be observed that large data-lengths decelerates the identification process but result in good estimations compared to small data-lengths. For the IBB
6.2. Example 2: Comparative study

Since no prior identification methods for FO Hammerstein models with input dead-zone exists, the comparison is performed whereby the linear subsystem was modeled with an IO. This serves to display the advantages of modeling with FO. Consider the Hammerstein system in [19] whereby the linear subsystem is given by the difference equation

\[ y_t = (0.1346q^{-1} - 0.1158q^{-2})v_t \]  

(65)

where \( q^{-1} \) is the unit delay operator, and the dead-zone function is

\[ v_t = \begin{cases} 1.2(u_t - 1.2) & u_t > 1.2 \\ 0 & -1 \leq u_t \leq 1.2 \\ u_t + 1 & u_t < -1. \end{cases} \]  

(66)

where \( m_r = 1.2, m_I = 1, d_r = 1.2 \) and \( d_I = -1 \). Noting that the signals were sampled by a sampling period of 3 s, the difference equation can be transformed into the continuous time second-order transfer function

\[ G(s) = \frac{0.05s + 0.0025}{s^2 + 0.125s + 0.0025}e^{-0.5s}. \]  

(67)

The system was excited with the same test input as Example 1. The data-length \( F \) was set to 512 samples. The identification results are tabulated in Table 4 displaying comparison of the proposed method with two other established identification methods. The dead-zone functions and output responses of the system are compared in Fig. 10a and Fig. 10b respectively. For the convenience, the response is shown for only one period (12 s) of the periodic input \( u_2 \).

The proposed identification technique accurately models the Hammerstein system with a reduced single pole FO time delay linear model, and identifies the input dead-zone parameters with very little error between the actual system and the estimated model response. The proposed method also has a lower identification error than the RI method. The identification error using the NI method is significantly lower than the proposed, however, is should be highlighted that in the NI method, the order of the linear subsystem is known prior, which is not the case for the proposed method, as the order of the fractional linear subsystem is estimated with the other linear parameters. Furthermore, it should be noted that the proposed scheme, in comparison, accurately identifies the system with reduced number of parameters.

6.2.1. Addition of time delay

The system under consideration can be modified to include a time delay in the input to the linear subsystem to validate the proposed method. Consider the linear subsystem in (67) but with a time delay of 0.5 s,

\[ G(s) = \frac{0.05s + 0.0025}{s^2 + 0.125s + 0.0025}s^{-0.5s}. \]  

(68)
Table 4
Results for Example 2.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \hat{m}_r )</td>
<td>1.2085</td>
<td>1.1954</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{m}_l )</td>
<td>1.0022</td>
<td>0.9874</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{d}_r )</td>
<td>-0.9954</td>
<td>-1.065</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{b}_r )</td>
<td>-</td>
<td>0.0549</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{b}_l )</td>
<td>0.6859</td>
<td>0.0037</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{a}_r )</td>
<td>-</td>
<td>1(^a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{a}_l )</td>
<td>13.7700</td>
<td>0.0009</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\beta}_r )</td>
<td>-</td>
<td>1(^a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\beta}_l )</td>
<td>-</td>
<td>2(^a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \hat{\alpha}_r )</td>
<td>1.0125</td>
<td>1(^a)</td>
</tr>
<tr>
<td>MSE((\times 10^{-5}))</td>
<td></td>
<td>1.4529</td>
<td>5.0486</td>
<td>0.0006(^b)</td>
</tr>
</tbody>
</table>

RI: = Recursive Identification; NI: = Non-Iterative Identification.

\(^a\)Fixed parameters.

\(^b\)Very low error due to commensurate order of linear dynamics.

Fig. 10. Estimation results — Example 2.

Table 5
Results with time delay.

<table>
<thead>
<tr>
<th>Model estimates</th>
<th>( \delta )</th>
<th>( \delta )</th>
<th>( \delta )</th>
<th>( \delta )</th>
<th>( \delta )</th>
<th>( \delta )</th>
<th>( \delta )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.2102</td>
<td>1.0034</td>
<td>1.1926</td>
<td>-0.9873</td>
<td>0.6752</td>
<td>13.5642</td>
<td>1.0154</td>
<td>0.5220</td>
</tr>
<tr>
<td>MSE = 1.6455 (\times 10^{-5})</td>
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The dead-zone function is kept as (66) and using the proposed method, the identification results are shown in Table 5. The very small MSE of the overall response validates the functionality of proposed method for time delay estimation.

6.2.2. Identification with double pole linear model

The Hammerstein system can be accurately estimated using a reduced single pole FO linear model. However, in accordance with Remark 10, the flexibility of the proposed algorithm to increase the model order of the linear subsystem can be verified. For instance, the linear subsystem can be modeled as a double pole fractional transfer function i.e. \( n = 2 \) and \( m = 0 \), as given below

\[
G(s^\alpha) = \frac{b_0}{a_2 s^{2\alpha} + a_1 s^\alpha + 1 + e^{-\tau L}}
\]

and (62) becomes

\[
\begin{align*}
A(\cdot) &= a_2 I + a_1 B_{22-a_1} + B_{22-a_0} \\
B(\cdot) &= b_0 B_{a_2}
\end{align*}
\]

The identification results in this case are displayed in Table 6.
It is seen that in this case, modeling with a double pole increases the accuracy. However, the number of parameters to be estimated is higher. The estimation of the dead-zone parameters is also slightly more accurate when using a double pole model. Therefore, for the IBB strategy, there is a trade-off between accuracy and number of parameters to be estimated. It is to be noted that increasing the order of the model results in an increase of parameters to be estimated, which directly affects the computation load of the identification. It is suggested to use the single pole model for estimation if computation load is factor for consideration (most likely scenario in a practical situation), otherwise higher order models can be used for improved accuracy.

6.3. Example 3: Practical study [DC Servo Motor]

The GSMT2014© DC Servo Control Trainer was utilized for the experiment. The GSMT2014© comprises of two twin motors based on a motion controller and an intelligent servo drive that enables real time control via MATLAB©. The general transfer function of the DC motor dynamic system is the relation of the motor revolving speed to the armature voltage and is expressed in the continuous domain as

$$G(s) = \frac{K_c}{\tau_m s + 1}$$  \hspace{1cm} (71)

where $K_c$ is the revolving speed constant and $\tau_m$ is the mechanical time constant. In accordance with user manual of the GSMT2014©, the parameters of the intelligent drive can be set and downloaded into the system using the software EasyMotion Studio©. The software allows the setup of the external reference and the dead-zone point and the dead-zone range can be set. This dead-zone is the relation of the motor speed reference to the armature voltage. The dead-zone point was set to $-1$ V and the dead-zone range was set to 2 V. The slopes $m_r, m_l$ cannot be set and are unknown. The dead-zone function thus can be written as,

$$v_t = \begin{cases} 
  m_r (u_t - 1) & u_t > 1 \\
  0 & -1 \leq u_t \leq 1 \\
  m_l (u_t + 1) & u_t < -1
\end{cases}$$  \hspace{1cm} (72)

The experimental setup is shown in Fig. 11. In order to provide the system with test input signal and measure the open-loop controller-free data, the system was interfaced with a Googol Technology Analog Control Box. The input test signal provided from a function generator and a single motor were connected to the power amplifier and motor connector modules of analog control box respectively, and the output measurement was taken. The input-output data was collected in two stages, first with the binary input and then the multi-sine input.

The data was taken for 100 s for each input signal and 50 s of measured data ($t_0 = 20$ s) was utilized for the identification. The system was excited with $u_{1_{\text{max}}} = 5$ V and $(u_{2_{\text{min}}}, u_{2_{\text{max}}}) = (-6, 6)$ V. The identification results are tabulated in Table 7. The estimated physical dead-zone nonlinearity in the DC servo motor is displayed in Fig. 12a and the actual output response and the estimated output response of the nonlinear system is compared for one period of input in Fig. 12b.

The proposed technique is practically applicable as the identified model accurately maps the response of the actual DC servo system. The physical dead-zone introduced in the system is also estimated, together with the input time delay. This estimation of the nonlinearity can allow principles of nonlinearity cancellation to be applied to linearize the system, permitting quality control.
The identification results also highlight the practicality of modeling with fractional differential models as an accurate model of the DC servo motor was estimated together with the input nonlinearity in the system from measured data.

7. Conclusion

The parametric identification of Hammerstein systems with input delay and dead-zone was successfully incorporated with fractional modeling techniques. The use of BPOMs allows the fractional differential equations to be converted to algebraic equations, permitting estimation of the immeasurable intermediate signals. The IBB strategy proposed utilizes these estimations to separately identify the nonlinearity and linear subsystem parameters, ensuring simultaneous identification of the fractional differentiation order(s), the time delay and the linear coefficients. The numerical example confirmed the convergence of the parameters during identification as well the robustness of the algorithm to measurement noise. The comparative study displayed the superiority of reported FO modeling and technique to classical IO models as successful identification was achieved with reduced number of parameters and lower computational load due to algorithm being on the basis of algebraic operations. And finally, a successful experiment with a DC servo motor exhibited the practicality and functionality of the suggested methodology.

CRediT authorship contribution statement

Vineet Prasad: Conception and design of study, Acquisition of data, Analysis and/or interpretation of data, Writing – original draft, Writing – review & editing. Utkal Mehta: Acquisition of data, Analysis and/or interpretation of data, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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