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A solution to the two-dimensional findpath problem

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Abstract. *The 'findpath problem', a well-known problem in robotics, is the problem of finding a path for a moving solid among other solid obstacles. In this paper, a solution is proposed for the two-dimensional case where two point masses are required to move to designated areas or targets located in the horizontal plane while avoiding moving or stationary planar objects. The main tool used to solve the problem is the 'second or direct method of Liapunov', a powerful mathematical tool usually associated with the stability analysis of nonlinear systems. The theory developed from solving the two-dimensional findpath problem is then applied to the problem of cooperation between two planar robot arms. Computer simulations show the effectiveness of the proposed method.*

1 General introduction

One of the most interesting theoretical undertakings in robotics research is the quest for solutions to a seemingly simple two-dimensional space geometric problem: 'given a robot and a description of its working space or *workspace*, propose a path that the robot can follow. In particular, if the workspace is cluttered with solid objects or obstacles, propose a collision-free path that can lead the mobile robot from the desired starting point to the desired location or *target*'. A complete solution to this problem must also take into account the generation of the shortest, smoothest and safest path among all the collision-free paths between the initial position and the target.

Researchers, over the years, have come up with several sophisticated algorithms for tackling this problem, appropriately called the 'robot path planning problem' or the 'findpath problem'. Applicable to mobile robots that, in theory, could be considered as solid objects such as circles or polygons in two-dimensional space, these algorithms can be grouped into two major categories (Günther & Azarm,

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1993; Sheu & Xue, 1993): (1) those that employ some kind of graph search techniques, and (2) those that employ some kind of physical analogy.

Basically, in a graph search technique, a collision-free path is generated through searching a graph formed out of straight lines that connect the origin and destination via the vertices of solid obstacles, or via patches of free space that have been decomposed into geometric primitives such as cones and cylinders. Theoretically, graph search techniques are elegant. However, in practice, they tend to be computationally intensive.

In the second category, various types of physical analogies are employed. An example involves the placing of position-dependent artificial potential fields with repulsive and attractive poles around obstacles and targets, and a collision-free path is determined by how much the robot is attracted to or repelled by the poles. The Laplace equation and a hydrodynamic analogy, which utilizes harmonic functions, have been used to establish the artificial potential fields. Another example utilizes the idea of following a 'scent' to the target. Mathematically, the scent could be represented by an unsteady diffusion equation.

Physical analogy-based methods have several advantages over the graph-based approaches, the most important being the easier implementation of the former in practice. However, one of the major drawbacks of the potential fields methods is the possibility of having a collision-free path leading not to the target but to 'traps' outside the target. These traps, like the target, are in fact local minima or points of zero potential and kinetic energy.

In this paper, we develop a technique that falls into the second category. In essence, a potential field method, the technique uses a method traditionally associated with nonlinear dynamical systems. It is called the 'second or direct method of Liapunov' and considered a powerful mathematical tool that can be used to analyze the stability properties of nonlinear systems. Outlined in the classical memoir "The general problem of the stability of motion" by A. M. Liapunov (or Lyapunov) in 1892 (Lyapunov Centenary Issue, 1992), the method is now also used in relatively new fields of research such as chaos, neurodynamics, and parallel computing (Skowronski, 1990).

In this paper, we tentatively follow a school of thought, initiated by Stonier (1990), that promotes the Liapunov method as a viable alternative to the available methods that solve the findpath problem.

2 Introduction

The 'second or direct method of Liapunov' or simply the 'Liapunov method or technique' (Lyapunov, 1892), is a generalization of two physical principles for conservative systems (Boyce & DiPrima, 1992), namely, (a) a rest position is stable if the potential energy is local minimum, otherwise it is unstable, and (b) the total energy is a constant during any motion. The method is central to understanding concepts that could reveal the stability nature of a nonlinear system. Traditionally associated with control theory, the method is now also used in relatively new domains such as chaos, neurodynamics, and parallel computing (Skowronski, 1990).

A recent application of the method is one that deals with the geometric problem of finding a collision-free path for a moving solid object among other solid objects. This problem, which is called the 'findpath problem', is well known in robotics and several ways of generating solutions have been proposed. Among the more theoretically elegant but computationally intensive algorithms are those that are based on some kind of graph search methods or geometric search algorithms such as the Voronoi polygon approach to the closest-point problems considered by Shamos and Hoey (1975), the piano movers' problem proposed by Schwartz and Sharir (1983), the configuration space method by Lozano-Pérez (1983), the free space decomposition approach by Brooks (1983), and the octree representation by Herman (1986). Basically, graph search methods seek to establish straight lines or connectivity graphs that allow obstacles to 'see' each other's position, shape, and orientation.

Simpler algorithms tend to use physical analogies. In 1986, a potential field method was proposed by Khatib. In his approach, position-dependent artificial potential fields are placed around obstacles and collision-free can be generated by determining the number of repulsive and attractive poles. Improvements to Khatib's method have been proposed by Connolly *et al.* (1990) who used the Laplace equation as a potential function which does not exhibit local minima, Tarassenko and Blake (1991) who treated the Laplace equation under Neumann boundary conditions, and Kim and Khosla (1991) who used a hydrodynamic analogy to establish an artifical potential field. Another physical analogy-based approach suggested by Günther and Azarm (1993) involves the use of unsteady diffusion equations to represent a gaseous substance that can be seeped into the environment. The level of concentration of the gaseous substance can then be used to determine the existence of obstacles and targets.

Although troubled with the existence of local minima traps, the speed and extensibility of the physical analogy-based algorithms (to higher dimensions) make them an excellent alternative approach to the findpath problem.

On a more practical side involving multiple robots working in a coordinated fashion, hierarchical strategies, such as those proposed by Chien *et al.* (1988) and Freund and Hoyer (1988), appear to offer an easy transition from the mathematical complexities usually associated with nonlinear systems to application.

In this paper, we develop the Liapunov method for the two-dimensional findpath problem involving the collision avoidance of two 'point masses' or 'point objects', and show that the physical analogy-based technique can provide a viable alternative to the available theoretical methods. Indeed, as first demonstrated by Stonier (1990) and then by Vanualailai *et al.* (1995), the method has been shown to provide, via 'Liapunov-like functions', nonlinear analytic forms of control laws for the planar movement of two point objects, moving to designated areas or 'targets' located in the horizontal plane among mobile and stationary solid objects. Stonier (1992) went further and applied his method to the problem of control of two planar robot arms.

We begin our discussion by briefly stating the Liapunov method and then using the notations and basic methods of constructing targets and obstacles devised by Stonier (1990), we provide a simple dynamical model of the two point masses moving in two-dimensional space.

There are two major differences between this paper and those of Stonier. The first is that our approach throughout this paper is more rigorous in that a single 'Liapunov function' is precisely developed for the entire system, a departure from Stonier's intuitive approach which uses different Liapunov-like functions for different components of the system. The second major difference concerns Stonier's use of the so-called 'right-of-way' assumption, which allows one object to register the position of the other as a constant in a sufficiently small time interval before making a move. The assumption has two shortcomings. The first is the difficulty one may have in justifying the use of the components of the position vector at time t of the system trajectory as constants in the Liapunov-like function. The second is the difficulty in the use of the assumption in a multi-point system where the assumption poses the problem of deciding which object or objects should be held in a given time interval. These problems were overcome in Vanualailai *et al.* (1995) where a single Liapunov-like function for the entire system, instead of a Liapunov-like function for each point mass, is constructed.

The two papers however have a common drawback. The Liapunov-like functions defined in them do not satisfy the Liapunov stability condition that they should be precisely zero at stable equilibrium points of the system. The attempt to satisfy this condition in both papers saw a restriction placed on one of the two types of parameters associated with the Liapunov method for the findpath problem. The two parameters are the 'control' and 'convergence parameters', and the restriction is in the requirement that the control parameters, which help in obtaining the desired trajectory, be sufficiently small so that the existence of a stable equilibrium state of the system in a neighborhood of the center of a target could be guaranteed. In other words, with this restriction, we get the best possible result at the end of a trajectory, and that is, an object ceases motion very close to the center of its target. Indeed, in applying his technique to the control of two manipulators, Stonier (1992) requires another method of trajectory planning to place the gripper of a manipulator precisely at the center of its target.

In the first part of this paper (Section 3), this problem of not reaching the center of a target is solved once and for all by the use of a function that satisfies the sufficient conditions of Liapunov's stability theorem. This Liapunov function can be easily extended to encompass multi-point systems, and it requires the control parameters only for the purpose of controlling the direction of the trajectory. Moreover, the proposed function need not be generalized, as in the case in Vanualailai *et al.* (1995), to obtain safe and smooth trajectories.

In the second part (Section 4), we apply our method to the problem of coordination between two planar robot arms.

3 A two-point system

3.1 Stability via the Liapunov function

Consider the autonomous nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \qquad t_0 \geqslant 0 \tag{1}$$

in which $\mathbf{f}: \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$ is assumed to be smooth enough to guarantee existence, uniqueness, and continuous dependence of solutions $\mathbf{x}(t) = \mathbf{x}(t, t_0, \mathbf{x}_0)$ of (1) in Ω, an open set in $\mathbf{R}^n$.

For the purpose of considering stability concepts in the sense of Liapunov, we assume that $\mathbf{f}(\mathbf{0}) \equiv \mathbf{0}$ so that $\mathbf{x}(t) \equiv \mathbf{0}$ is the 'equilibrium state' of (1) through $\mathbf{0}$ in Ω for all $t \geqslant t_0$.

Definition 1. The equilibrium state is stable if, for each $\varepsilon > 0$ and $t_0 \geqslant 0$, there is a $\delta > 0$ such that $\|\mathbf{x}_0\| < \delta$ and $t \geqslant t_0$ imply $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon$.

The direct method of Liapunov states that this equilibrium state is stable if, in a neighborhood D of the equilibrium state, there exists a real scalar function V such that:

(a) $V(\mathbf{0}) = 0$,
(b) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and
(c) $\dot{V}_{(1)}(\mathbf{x}) = \Sigma_{i=1}^{n} (\partial V/\partial x_i) f_i(\mathbf{x}) \leqslant 0$ for all $\mathbf{x} \in D$, where x_i and f_i, $i = 1, 2, 3, \ldots, n$, are respectively the components of $\mathbf{x}$ and $\mathbf{f}$.

When V successfully meets the above conditions, it is called the 'Liapunov function' for system (1).

3.2 Dynamics of two-point objects

Consider the system of ordinary differential equations (ODEs)

$$\left.\begin{aligned} \dot{x}_1 &= x_2, & \dot{y}_1 &= y_2 \\ \dot{x}_2 &= u_1, & \dot{y}_2 &= v_1 \\ \dot{x}_3 &= x_4, & \dot{y}_3 &= y_4 \\ \dot{x}_4 &= u_2, & \dot{y}_4 &= v_2 \end{aligned}\right\} \tag{2}$$

In the z_1–z_3 plane, we refer to the point (x_1, x_3) as 'Object 1' and (y_1, y_3) as 'Object 2'. System (2) is therefore a description of the instantaneous velocities $(\dot{x}_1, \dot{x}_3) = (x_2, x_4)$ and $(\dot{y}_1, \dot{y}_3) = (y_2, y_4)$ and the instantaneous accelerations $(\dot{x}_2, \dot{x}_4) = (u_1, u_2)$ and $(\dot{y}_2, \dot{y}_4) = (v_1, v_2)$ of the point objects.

We shall use the vector notations

$$\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4, \qquad \mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbf{R}^4$$

to refer to the position and velocity components of Object 1 and Object 2, respectively.

Let us assume that we can transfer the point objects from one point to another in the z_1–z_3 plane by adjusting the accelerations appropriately. That is, via (u_1, u_2) and (v_1, v_2), we have the ability to control the direction and speed of the point objects to their respective targets.

The targets are defined as circular regions in the z_1–z_3 plane enclosing a set of fixed points. Thus, if we let (p_1c_1, p_1c_2) and (p_2c_1, p_2c_2) be the centers, and rp_1 and rp_2 be the radii of the targets, then the sets

$$T_1 = \{(z_1, z_3) \in \mathbf{R}^2 \colon (z_1 - p_1c_1)^2 + (z_3 - p_1c_2)^2 \leqslant rp_1{}^2\}$$

$$T_2 = \{(z_1, z_3) \in \mathbf{R}^2 \colon (z_1 - p_2c_1)^2 + (z_3 - p_2c_2)^2 \leqslant rp_2^2\}$$

become the 'fixed target sets', with T_1 being the target set of Object 1 and T_2, the target set of Object 2. The targets can also become 'fixed antitarget sets' in the sense that T_1 is the fixed obstacle of Object 2, and T_2 is the fixed obstacle of Object 1. To express this mathematically, we write

$$AT_1^2 = T_1 \qquad \text{and} \qquad AT_1^1 = T_2$$

where the superscript indicates which point object is being considered, and the subscript indicates which obstacle it is. Hence, AT_1^2 is antitarget number 1 of Object 2, and AT_1^1 is antitarget number 1 of Object 1.

Next, we place two point objects in circular 'secure avoidance regions' centered at the point objects themselves with radii rap_1 and rap_2. Thus, the sets

$$AT_2^1(t) = \{(z_1, z_3) \in \mathbf{R}^2 \colon [z_1 - y_1(t)]^2 + [z_3 - y_3(t)]^2 \leqslant rap_2^2\}$$

$$AT_2^2(t) = \{(z_1, z_3) \in \mathbf{R}^2 \colon [z_1 - x_1(t)]^2 + [z_3 - x_3(t)]^2 \leqslant rap_1^2\}$$

are the 'moving antitarget sets' of Object 1 and Object 2, respectively.

We can now state the control objectives as follows:

[O1] To control the movement of Object 1 to its fixed target T_1 while ensuring it avoids the fixed antitarget $AT_1^1 = T_2$ and the moving antitarget $AT_2^1(t)$.

[O2] To control the movement of Object 2 to its fixed target T_2 while ensuring it avoids fixed antitarget $AT_1^2 = T_1$ and the moving antitarget $AT_2^2(t)$.

3.3 Obstacle avoidance and target attraction

Geometrically, in three-dimensional space, the Liapunov function V looks like a parabolic 'mirror' pointing upward or a 'cup' on a table (LaSalle & Lefschetz, 1961). In two-dimensional space, the cup's circular level curves represent loci for constant energy, which as time passes, must shrink to, but not necessarily reach, the point representing the bottom of the cup or the minimum value of the Liapunov function V. Thus, the motion along the phase trajectory takes place in the direction of decreasing V loci.

In our scheme, for a point object, we consider a cup-shaped surface and let the bottom of the cup be the center of the target.

3.3.1 Object 1.

Attraction to target. For the attraction to the target

$$T_1 = \{(z_1, z_3) \in \mathbf{R}^2 \colon (z_1 - p_1 c_1)^2 + (z_3 - p_1 c_2)^2 \leqslant rp_1^2\}$$

we consider the function

$$V_0(\mathbf{x}) = \frac{1}{2}[(x_1 - p_1 c_1)^2 + (x_3 - p_1 c_2)^2 + x_2^2 + x_4^2]$$

which is a measure of the distance from Object 1 to the target and the speed of Object 1. Once we have established an appropriate Liapunov function for system (2), V_0 would act as an attractor by having Object 1 move down its decreasing closed surfaces of constants to the target centered at (p_1c_1, p_1c_2). As we shall see later, the inclusion of the velocity components will help in the formulation of a control law that provides for a 'damping' capability that determines the rate of convergence of Object 1 to its target T_1.

Note that $V_0(p_1c_1, 0, p_1c_2, 0) = 0$ and $V_0(\mathbf{x}) > 0$ for all $\mathbf{x} \neq (p_1c_1, 0, p_1c_2, 0)$.

Avoidance of the fixed obstacle. For the avoidance of the fixed obstacle

$$AT_1^1 = T_2 = \{(z_1, z_3) \in \mathbf{R}^2 \colon (z_1 - p_2 c_1)^2 + (z_3 - p_2 c_2)^2 \leqslant rp_2^2\}$$

we consider the function

$$V_1(\mathbf{x}) = \frac{1}{2}[(x_1 - p_2 c_1)^2 + (x_3 - p_2 c_2)^2 - rp_2^2]$$

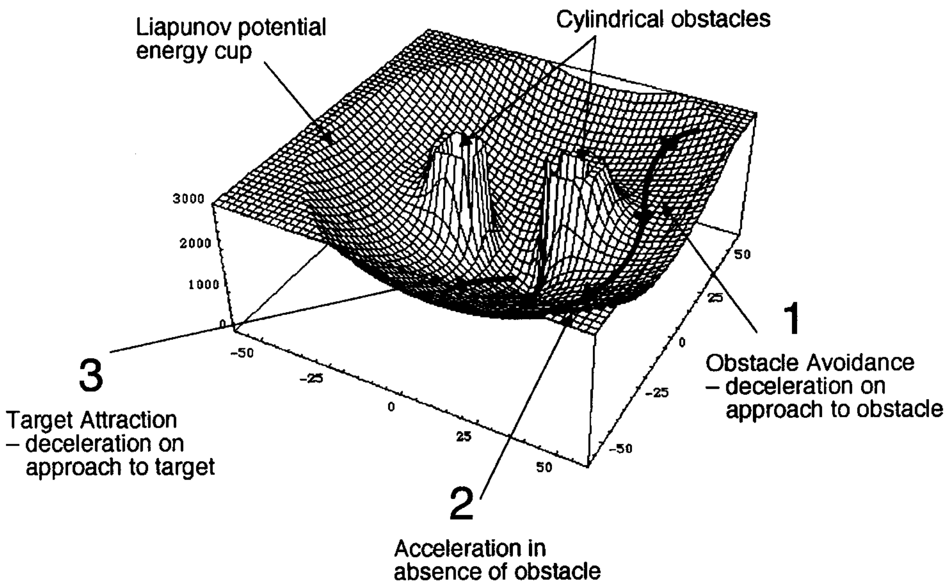

Fig. 1. Obstacle avoidance scheme via the Liapunov function.

a measure of the distance from Object 1 to the target T_2 of Object 2, with $V_1(\mathbf{x}) > 0$ over the domain $\{\mathbf{x} \in \mathbf{R}^4: (x_1 - p_2 c_1)^2 + (x_3 - p_2 c_2)^2 > rp_2^2\}$.

In three-dimensional space, the surface

$$s_1 = \frac{c}{(x_1 - p_2 c_1)^2 + (x_3 - p_2 c_2)^2 - rp_2^2}, \qquad c = \text{constant} > 0$$

is a right circular cylinder with radius rp_2. If this cylinder is a part of the Liapunov potential energy cup as illustrated in Fig. 1, then Object 1, in an intuitive sense, will naturally slow down as it reaches the saddle-like base of this structure and then avoid the structure as it sinks to the bottom of the cup. Since the point object must inevitably be attracted to the target set (this is the inherent advantage of the Liapunov function) and $(x_1 - p_2 c_1)^2 + (x_3 - p_2 c_2)^2 \to rp_2^2$ would imply an increase in energy (that is, $|s_1| \to +\infty$), we cannot have the situation where $(x_1 - p_2 c_1)^2 + (x_3 - p_2 c_2)^2 = rp_2^2$. Hence, in a term in the Liapunov function to be proposed, we can have V_1 appear in the denominator.

Avoidance of the moving obstacle. For the avoidance of the moving obstacle

$$AT_2^1(t) = \{(z_1, z_3) \in \mathbf{R}^2: [z_1 - y_1(t)]^2 + [z_3 - y_3(t)]^2 \leqslant rap_2^2\}$$

we consider the function

$$V_2(\mathbf{x}, \mathbf{y}) = \frac{1}{2}[(x_1 - y_1)^2 + (x_3 - y_3)^2 - rap_2^2]$$

a measure of the distance from Object 1 to the secure avoidance region $AT_2^1(t)$ about Object 2. We note that $V_2(\mathbf{x}, \mathbf{y}) > 0$ over the domain $\{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^4 \times \mathbf{R}^4: (x_1 - y_1)^2 + (x_3 - y_3)^2 > rap_1^2\}$.

If we let y_1 and y_3 be some constants, say k_1 and k_2, respectively, then in three-dimensional space, the surface

$$s_2 = \frac{c}{(x_1 - k_1)^2 + (x_3 - k_3)^2 - rap_2^2}, \qquad c = \text{constant} > 0$$

is also a right circular cylindrical body with radius rap_2. Thus, in the Liapunov function to be proposed, we want V_2 to appear in the denominator as well.

Obstacle avoidance and target attraction. A Liapunov function for system (2) must become zero once the last remaining object reaches the center of its target. For this purpose, we introduce, for Object 1, the function

$$F(\mathbf{x}) = \frac{1}{2}[(x_1 - p_1c_1)^2 + (x_3 - p_1c_2)^2] \geqslant 0 \; \forall \, \mathbf{x} \in \mathbf{R}^4$$

with $F(p_1c_1, 0, p_1c_2, 0) = 0$.

Now, for the intention of satisfying the above intuitive arguments that require V_1 and V_2 to be in the denominator in the Liapunov function to be proposed, we introduce constants $\beta_{11} > 0$ and $\beta_{12} > 0$. Using these constants, we can thus define

$$V(\mathbf{x}, \mathbf{y}) = V_0 + F \sum_{k=1}^{2} \frac{\beta_{1k}}{V_k}$$

for attraction and obstacle avoidance for Object 1. The function V is defined, continuous and positive over the domain

$$D(V) = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^4 \times \mathbf{R}^4 : V_1(\mathbf{x}) > 0, \; V_2(\mathbf{x}, \mathbf{y}) > 0\}$$

We can make a similar intuitive argument for Object 2.

3.3.2 Object 2. For Object 2, we consider the functions

$$W_0(\mathbf{y}) = \frac{1}{2}[(y_1 - p_2c_1)^2 + (y_3 - p_2c_2)^2 + y_2^2 + y_4^2]$$

$$W_1(\mathbf{y}) = \frac{1}{2}[(y_1 - p_1c_1)^2 + (y_3 - p_1c_2)^2 - rp_1^2]$$

$$W_2(\mathbf{x}, \mathbf{y}) = \frac{1}{2}[(x_1 - y_1)^2 + (x_1 - y_3)^2 - rap_1^2]$$

for the attraction to the center of target T_2, the avoidance of the fixed antitarget $AT_1^2 = T_1$ and the avoidance of the moving antitarget $AT_2^2(t)$, respectively. To ensure that the Liapunov function to be proposed next will be zero at the center of targets, we consider, for Object 2

$$G(\mathbf{y}) = \frac{1}{2}[(y_1 - p_2c_1)^2 + (y_3 - p_2c_2)^2]$$

Then, introducing the constants $\beta_{21} > 0$ and $\beta_{22} > 0$, we form

$$W(\mathbf{x}, \mathbf{y}) = W_0 + G \sum_{k=1}^{2} \frac{\beta_{2k}}{W_k}$$

for target attraction and obstacle avoidance for Object 2. This function is defined, continuous and positive over the domain

$$D(W) = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^4 \times \mathbf{R}^4 : W_1(\mathbf{x}) > 0,\ W_2(\mathbf{x}, \mathbf{y}) > 0\}$$

3.4 Liapunov function

Consider as a tentative Liapunov function for system (2)

$$L(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}, \mathbf{y}) + W(\mathbf{x}, \mathbf{y}) = V_0 + F \sum_{k=1}^{2} \frac{\beta_{1k}}{V_k} + W_0 + G \sum_{k=1}^{2} \frac{\beta_{2k}}{W_k}$$

which is defined, continuous and positive on the open domain $D(V) \cap D(W)$, with

$$L(p_1c_1, 0, p_1c_2, 0, p_2c_1, 0, p_2c_2, 0) = 0$$

where $(p_1c_1, 0, p_1c_2, 0, p_2c_1, 0, p_2c_2, 0) \in D(V) \cap D(W)$. Along a solution of system (2), we have

$$\dot{L}_{(2)} = \dot{V}_0 + \sum_{k=1}^{2} \beta_{1k} \left(\frac{V_k \dot{F} - F \dot{V}_k}{V_k^2} \right) + \dot{W}_0 + \sum_{k=1}^{2} \beta_{2k} \left(\frac{W_k \dot{G} - G \dot{W}_k}{W_k^2} \right)$$

Treating the velocity components separately, we obtain

$$\begin{aligned} \dot{L}_{(2)}(\mathbf{x}, \mathbf{y}) = {} & [u_1 + f_2(\mathbf{x}, \mathbf{y})]x_2 + [u_2 + f_4(\mathbf{x}, \mathbf{y})]x_4 \\ & + [v_1 + g_2(\mathbf{x}, \mathbf{y})]y_2 + [v_2 + g_4(\mathbf{x}, \mathbf{y})]y_4 \end{aligned}$$

where

$$\begin{aligned} f_2 = {} & (x_1 - p_1c_1) \times \left(1 + \sum_{k=1}^{2} \frac{\beta_{1k}}{V_k} \right) - (x_1 - p_2c_1) \times \frac{\beta_{11}F}{V_1^2} \\ & - (x_1 - y_1) \times \left(\frac{\beta_{12}F}{V_2^2} + \frac{\beta_{22}G}{W_2^2} \right) \end{aligned}$$

$$\begin{aligned} f_4 = {} & (x_3 - p_1c_2) \times \left(1 + \sum_{k=1}^{2} \frac{\beta_{1k}}{V_k} \right) - (x_3 - p_2c_2) \times \frac{\beta_{11}F}{V_1^2} \\ & - (x_3 - y_3) \times \left(\frac{\beta_{12}F}{V_2^2} + \frac{\beta_{22}G}{W_2^2} \right) \end{aligned}$$

$$\begin{aligned} g_2 = {} & (y_1 - p_2c_1) \times \left(1 + \sum_{k=1}^{2} \frac{\beta_{2k}}{W_k} \right) - (y_1 - p_1c_1) \times \frac{\beta_{21}G}{W_1^2} \\ & + (x_1 - y_1) \times \left(\frac{\beta_{12}F}{V_2^2} + \frac{\beta_{22}G}{W_2^2} \right) \end{aligned}$$

$$g_4 = (y_3 - p_2c_2) \times \left(1 + \sum_{k=1}^{2} \frac{\beta_{2k}}{W_k}\right) - (y_3 - p_1c_2) \times \frac{\beta_{21}G}{W_1^2}$$

$$+ (x_3 - y_3) \times \left(\frac{\beta_{12}F}{V_2^2} + \frac{\beta_{22}G}{W_2^2}\right)$$

Let there be constants $\rho_1\gamma_1 > 0$, $\rho_1\gamma_2 > 0$, $\rho_2\gamma_1 > 0$, and $\rho_2\gamma_2 > 0$, such that

$$-\rho_1\gamma_2 x_2 = u_1 + f_2(\mathbf{x}, \mathbf{y}), \qquad -\rho_1\gamma_4 x_4 = u_2 + f_4(\mathbf{x}, \mathbf{y})$$

$$-\rho_2\gamma_2 y_2 = v_1 + g_2(\mathbf{x}, \mathbf{y}), \qquad -\rho_2\gamma_4 y_4 = v_2 + g_4(\mathbf{x}, \mathbf{y})$$

Then

$$\dot{L}_{(2)} = -\rho_1\gamma_2 x_2^2 - \rho_1\gamma_4 x_4^2 - \rho_2\gamma_2 y_2^2 - \rho_2\gamma_4 y_4^2$$

and

$$u_1(\mathbf{x}, \mathbf{y}) = -f_2(\mathbf{x}, \mathbf{y}) - \rho_1\gamma_2 x_2 \tag{3}$$

$$u_2(\mathbf{x}, \mathbf{y}) = -f_4(\mathbf{x}, \mathbf{y}) - \rho_1\gamma_4 x_4 \tag{4}$$

$$v_1(\mathbf{x}, \mathbf{y}) = -g_2(\mathbf{x}, \mathbf{y}) - \rho_2\gamma_2 y_2 \tag{5}$$

$$v_2(\mathbf{x}, \mathbf{y}) = -g_4(\mathbf{x}, \mathbf{y}) - \rho_2\gamma_4 y_4 \tag{6}$$

giving the control laws $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ for the planar movement of Object 1 and Object 2, respectively.

Finally, if we let

$$\mathbf{x}_e = (p_1c_1, 0, p_1c_2, 0) \in \mathbf{R}^4 \quad \text{and} \quad \mathbf{y}_e = (p_2c_1, 0, p_2c_2, 0) \in \mathbf{R}^4$$

then we have $(\mathbf{x}, \mathbf{y}) = (\mathbf{x}_e, \mathbf{y}_e) \in D(V) \cap D(W)$ as an equilibrium state of system (2).

We have thus the following properties of L:

(i) $L(\mathbf{x}, \mathbf{y})$ is continuous and has first partial derivatives in the region $D(V) \cap D(W)$ in the neighborhood of the stable equilibrium state $(\mathbf{x}_e, \mathbf{y}_e)$,
(ii) $L(\mathbf{x}_e, \mathbf{y}_e) = 0$,
(iii) $L(\mathbf{x}, \mathbf{y}) > 0 \ \forall \ (\mathbf{x}, \mathbf{y}) \in D(V) \cap D(W) \backslash (\mathbf{x}_e, \mathbf{y}_e)$,
(iv) $\dot{L}_{(2)}(\mathbf{x}, \mathbf{y}) \leqslant 0 \ \forall \ (\mathbf{x}, \mathbf{y}) \in D(V) \cap D(W)$.

Hence, L is a Liapunov function for system (2).

The following theorem summarizes the above discussions:

Theorem 1. The equilibrium state $(\mathbf{x}_e, \mathbf{y}_e)$ of system (2) is stable provided u_1, u_2, v_1, and v_2 are defined as (3), (4), (5), and (6), respectively.

Geometrically, if we take all variables except x_1 and x_2 as some constants, and plot $z = L(\mathbf{x}, \mathbf{y})$ as a function of (x_1, x_3), say $z = L(x_1, x_3)$, then we have Fig. 2, which shows clearly the two obstacles that Object 1 has to avoid. Similarly, $z = L(y_1, y_3)$ will show the two obstacles that Object 2 has to avoid.

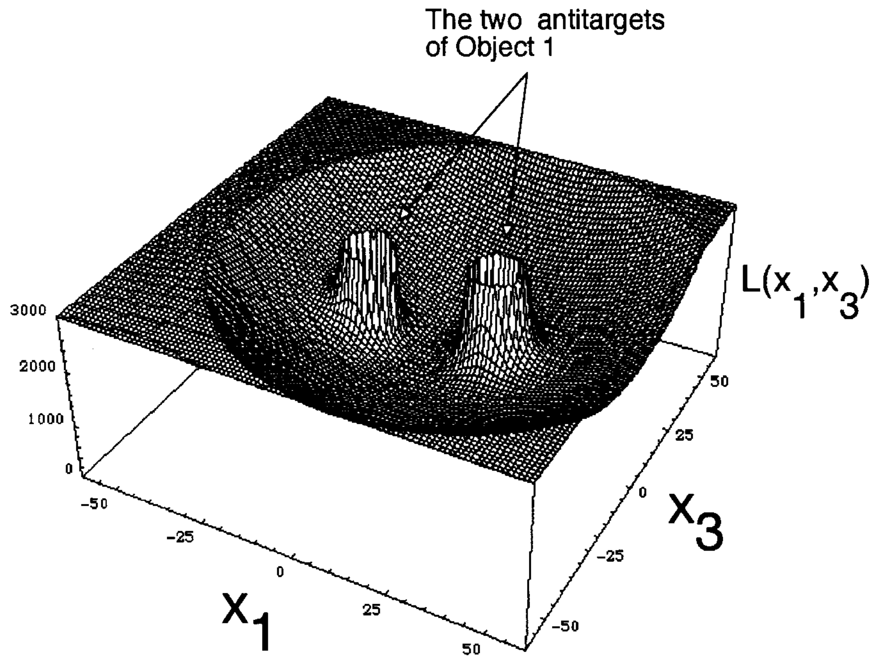

Fig. 2. The two obstacles of Object 1.

3.5 Control and convergence parameters

3.5.1 *Control parameters.* Because

$$L(\mathbf{x}, \mathbf{y}) = V_0 + F \sum_{k=1}^{2} \frac{\beta_{1k}}{V_k} + W_0 + G \sum_{k=1}^{2} \frac{\beta_{2k}}{W_k}$$

is a Liapunov function, we are at liberty to increase or decrease the parameters β_{1k} and β_{2k} as much as we please to obtain a desired trajectory without worrying whether an object will reach the center of its target. In Stonier (1990) and Vanualailai *et al.* (1995), there is no function that does the work of F or G and therefore one must rely solely on the sizes of β_{1k} and β_{2k} to ensure that an object approaches the center of its target. That is, the absence of these functions causes the objects to cease motion near to, but not at the center of their targets, requiring therefore the condition that β_{1k} and β_{2k} be sufficiently small. In our case, a change in the values of β_{1k} and β_{2k} simply affects the shape of the cup L to give the appropriate trajectories.

For example, if Object 1 starts closer to an obstacle, say T_2, then V_1 is larger, and hence L is at a larger locus for constant energy. This means Object 1 is subjected to a larger repulsive force which can cause it to initially repel violently from T_2. Decreasing β_{11} will reduce the effect of this repulsion. If, on the other hand, Object 1 barely avoids the stationary obstacle T_2, then an increase in β_{11} enhances the effect of V_1, resulting in Object 1 avoiding T_2 relatively earlier.

Other examples are if we need more leeway between the moving point objects, then we can increase either β_{11} or β_{21}, or both, and if Object 2 avoids the stationary obstacle T_1 from too large a distance, then a decrease in β_{22} is required (this could also reduce the time of arrival).

Whichever the situation, and barring the existence of stable critical points or saddle points outside the targets, the trajectories lead to the centers precisely.

Hence, in our method, the parameters β_{1k} and β_{2k}, which we may call control parameters, are the major factors in determining the amounts of $\mathbf{u}$ and $\mathbf{v}$ needed to give us a satisfactory 'trajectory control' of the objects to the center of their targets.

3.5.2. Convergence parameters. The controllers $\mathbf{u}$ and $\mathbf{v}$ can provide us with the information on the rate of convergence of the objects to their targets. For on letting

$$r_2(t, x_1, x_3, y_1, y_3) = -f_2(\mathbf{x}, \mathbf{y}), \qquad r_4(t, x_1, x_3, y_1, y_3) = -f_4(\mathbf{x}, \mathbf{y})$$

$$p_2(t, x_1, x_3, y_1, y_3) = -g_2(\mathbf{x}, \mathbf{y}), \qquad p_4(t, x_1, x_3, y_1, y_3) = -g_4(\mathbf{x}, \mathbf{y})$$

we have, given the initial states at time $t_0 \geqslant 0$

$$\mathbf{x}_0 = (x_1(t_0), x_2(t_0), x_3(t_0), x_4(t_0))$$

$$\mathbf{y}_0 = (y_1(t_0), y_2(t_0), y_3(t_0), y_4(t_0))$$

and knowing that

$$(\dot{x}_2, \dot{x}_4) = (u_1, u_2), \qquad (\dot{y}_2, \dot{y}_4) = (v_1, v_2)$$

the instantaneous velocity components

$$x_2(t) = \mathrm{e}^{-\rho_1\gamma_2(t-t_0)} x_2(t_0) + \int_{t_0}^{t} \mathrm{e}^{-\rho_1\gamma_2(t-s)} r_2(s, x_1, x_3, y_1, y_3)\, \mathrm{d}s$$

$$x_4(t) = \mathrm{e}^{-\rho_1\gamma_4(t-t_0)} x_4(t_0) + \int_{t_0}^{t} \mathrm{e}^{-\rho_1\gamma_4(t-s)} r_4(s, x_1, x_3, y_1, y_3)\, \mathrm{d}s$$

$$y_2(t) = \mathrm{e}^{-\rho_2\gamma_2(t-t_0)} y_2(t_0) + \int_{t_0}^{t} \mathrm{e}^{-\rho_2\gamma_2(t-s)} p_2(s, x_1, x_3, y_1, y_3)\, \mathrm{d}s$$

$$y_4(t) = \mathrm{e}^{-\rho_2\gamma_4(t-t_0)} y_2(t_0) + \int_{t_0}^{t} \mathrm{e}^{-\rho_2\gamma_4(t-s)} p_4(s, x_1, x_3, y_1, y_3)\, \mathrm{d}s$$

which clearly show that large values of $\rho_1\gamma_2$, $\rho_1\gamma_4$, $\rho_2\gamma_2$, and $\rho_2\gamma_4$ will decrease the rate of convergence of an object to its target. Unsuitable choices of $\rho_1\gamma_2$, $\rho_1\gamma_4$, $\rho_2\gamma_2$, and $\rho_2\gamma_4$, which we may call convergence parameters, can give us a rate of convergence that could result in an object converging too soon to its target, thus making it difficult to avoid an obstacle, or a rate of convergence that could be too slow to be of practical use.

In Section 3.3, we mentioned damping. Clearly, the convergence parameters are indeed the damping required to increase or reduce the speed of an object to its target.

Table 1. Example 1

Time interval	$[0, 30]$
RK4 step size	0.03
Target centers	$(p_1c_1, p_1c_2) = (12.0, 0.0)$, $(p_2c_1, p_2c_2) = (-12.0, 0.0)$
Target/antitarget radii	$rp_1 = rp_2 = 6.0$, $rap_1 = rap_2 = 6.0$
Initial states	$\mathbf{x}_0 = (-25.0,\ 1.0,\ 0.0,\ 5.0)$, $\mathbf{y}_0 = (25.0,\ 1.0,\ 0.0,\ 5.0)$
Control parameters	$\beta_{11} = \beta_{12} = 5.0$, $\beta_{21} = \beta_{22} = 5.0$
Convergence parameters	$\rho_1\gamma_2 = \rho_1\gamma_4 = 5.0$, $\rho_2\gamma_2 = \rho_2\gamma_4 = 5.0$

Remark. We have shown that, through the implementation of the control functions, system (2) is only stable. Therefore, we should expect that for some initial conditions, Object 1 and Object 2 will cease motion at other stable critical points or saddle points before they reach their targets. The easiest way to overcome this problem is to try out different values of the convergence and control parameters. A much more difficult assignment is to construct a Liapunov function that guarantees asymptotic stability, or better still, global asymptotic stability. This is an open problem.

3.6 Simulations

Example 1. In this example (see Table 1), the shortest path leading Object 1 to T_1 is blocked by T_2, and the shortest path leading Object 2 to T_2 is blocked by T_1. To illustrate collision avoidance between the moving obstacles, the control and convergence parameters are chosen such that the moving point objects are forced to approach each other.

Figure 3 clearly shows smooth, collision-free paths.

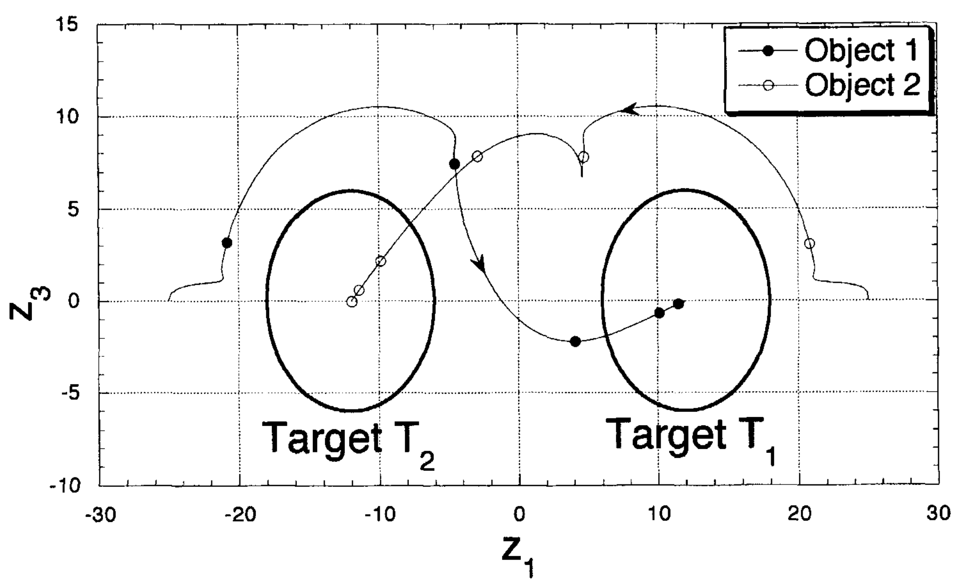

Fig. 3. For each point object, the shortest path leading to the target is blocked by an obstacle (Example 1).

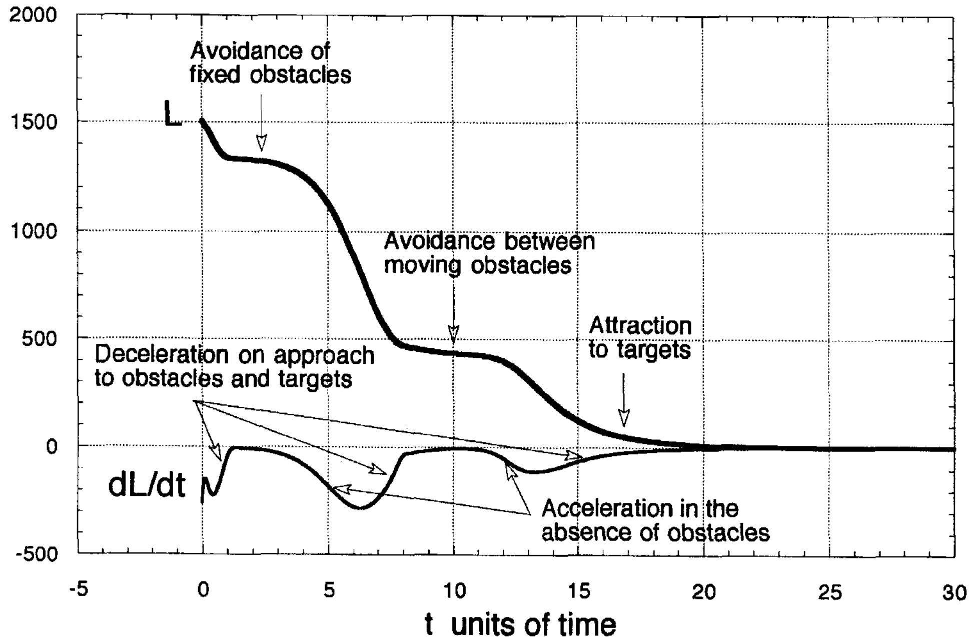

Fig. 4. The Liapunov function and its rate of change can tell how the point objects approach obstacles and targets (Example 1).

To see the nature of the Liapunov function $L(\mathbf{x}, \mathbf{y})$ and its effects on the point objects, we refer to Fig. 4. There, we can see clearly the decreasing nature of the function (hence, the cup in three-dimensional space). The almost flat regions indicate the period of collision avoidance. The instantaneous rates of change of L clearly indicate where the obstacles have accelerated or decelerated.

Finally, Fig. 5 shows the asymptotic behavior of the controllers. Note that Object 1 reaches its target in 22.6 units of time and Object 2 does so in 24.6 units of time.

Example 2. This example (see Fig. 6 and Table 2) illustrates the effectiveness of the control and convergence parameters. Starting from the same initial states as in Example 1, we force Object 1 to move 'below' its obstacles by having Object 2 converging faster (make $\rho_2\gamma_2, \rho_2\gamma_4$ relatively smaller than $\rho_1\gamma_2, \rho_1\gamma_4$) and increasing the repulsive forces from the antitargets (make β_{21}, β_{22} relatively bigger than β_{11}, β_{12}).

4 Application to two planar robot arms

In a robotic path planning system, a mobile robot or a robot manipulator and obstacles in a work environment are represented as fixed-shape objects, say, circles or polygons, or shape-changeable objects such as a set of cuboids with different sizes (Sheu & Xue, 1993). The central idea of doing such representations is to replace a complex-shaped robot or obstacle by a simpler figure that captures the morphological features of the complex shape. Since the simplified figure will often be used to approximate the clearance of a complex-shape robot amidst obstacles, it will be assumed to enclose the robot, so as to provide a conservative approximation.

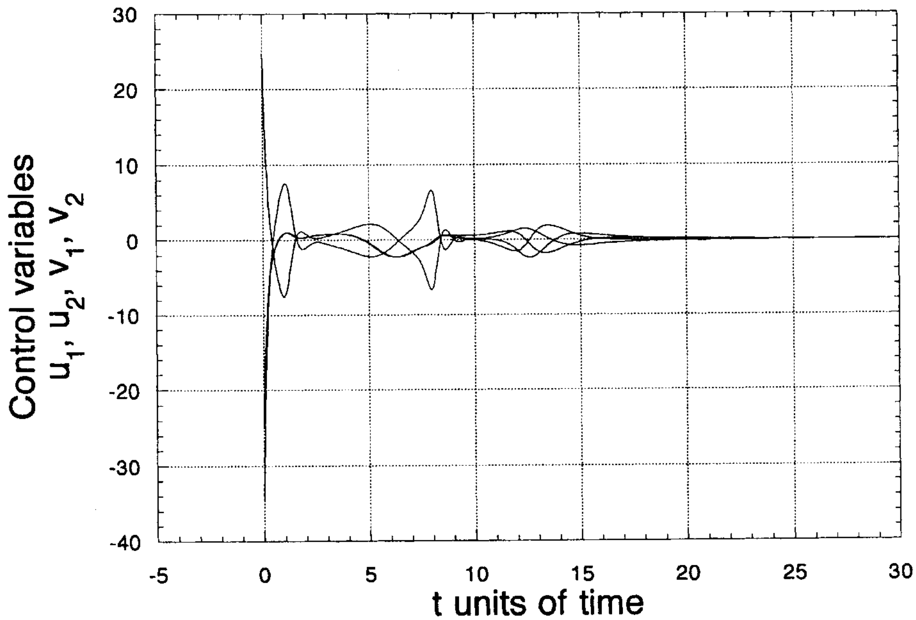

Fig. 5. Asymptotic behavior of the controllers (Example 1).

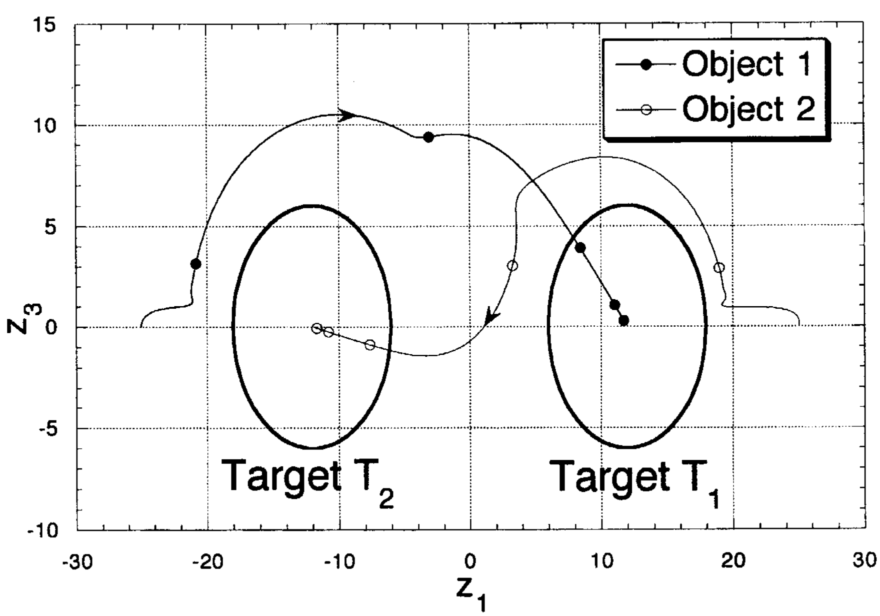

Fig. 6. Appropriate choices of the control and convergence parameters give collision-free smooth paths (Example 2).

Table 2. Example 2

Time interval	[0, 100]
Control parameters	$\beta_{11} = \beta_{12} = 5.0$, $\beta_{21} = \beta_{22} = 20.0$
Convergence parameters	$\rho_1\gamma_2 = \rho_1\gamma_4 = 20.0$, $\rho_2\gamma_2 = \rho_2\gamma_4 = 10.0$

The above observations fit well into the proposed Liapunov method, which works suitably with circular figures or ellipses. In our approach, therefore, we will place all moving or stationary objects in circles, and then trace a path for the desired object.

In our example, we will look at the simple planar robot arm described in Stonier (1990). However, instead of one, we will consider two robot arms working cooperatively in a common working environment. Moreover, we will use one Liapunov function for the entire system instead of two different Liapunov-like functions—one function for each arm as proposed in Stonier (1992).

The description of the robot arm conforms with those described in Freund and Hoyer (1988), namely, the robot arm has a translational joint and a rotational joint in the horizontal z_1–z_3 plane, and another translational joint vertical to the plane. The vertical movement is not considered in the avoidance strategies since in most practical cases it is restricted by such things as conveyer belts and assembly stands.

The arm consists of two links made up of uniform slender rods; the revolute first link with fixed length, and the prismatic second link which carries the payload at the gripper. It is assumed that the sliding motion of the second link relative to the first link is due to a linear torque (there is no rotation of the second link relative to the first). It is also assumed that the rotation of the manipulator is caused solely by an applied actuator torque and is parallel to the earth's surface so that gravity is not a factor.

As roughly shown in Fig. 7, the objective is to move the gripper from an initial position to a target, T_1, in the workspace, the accepted paths being, for example, smooth paths, Path 1 and Path 2.

With the help of Fig. 8 which shows a schematic representation of the arm in the horizontal z_1–z_3 plane, we assume that:

(a) the first link has a fixed length r_1,
(b) the manipulator has length $r(t)$ at time t,
(c) the manipulator has angular position $\theta(t)$ at time t,
(d) the manipulator has mass m_1 located at point A which is the center of mass,
(e) the payload of mass m_2 is located at the gripper at point B,
(f) the linear torque is $f_r(t)$ at time t, and
(g) the actuator torque is $\tau_0(t)$ at time t.

Using Lagrange's equations, it is easy to show that the equations of motion of the arm are

$$[m_1 r_1^2 + m_2 r^2(t)]\ddot{\theta}(t) + m_2 r(t)\dot{r}(t)\dot{\theta}(t) = \tau_0(t)$$

$$m_2\ddot{r}(t) - m_2 r(t)\dot{\theta}^2(t) = f_r(t)$$

The Liapunov method requires a state-space description of the equations of motion. Accordingly, let

$x_1 =$ the angular position, $\theta(t)$, of the arm

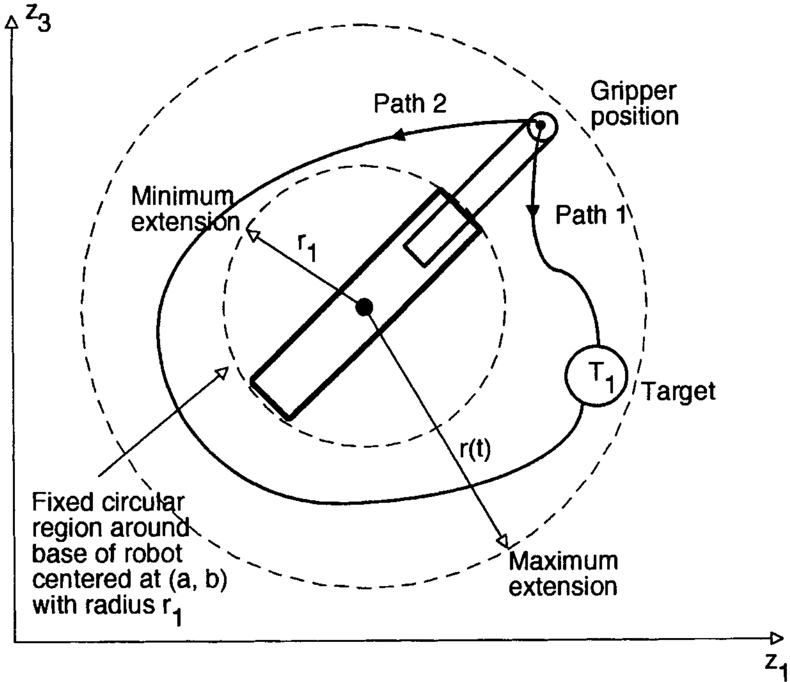

Fig. 7. A planar manipulator.

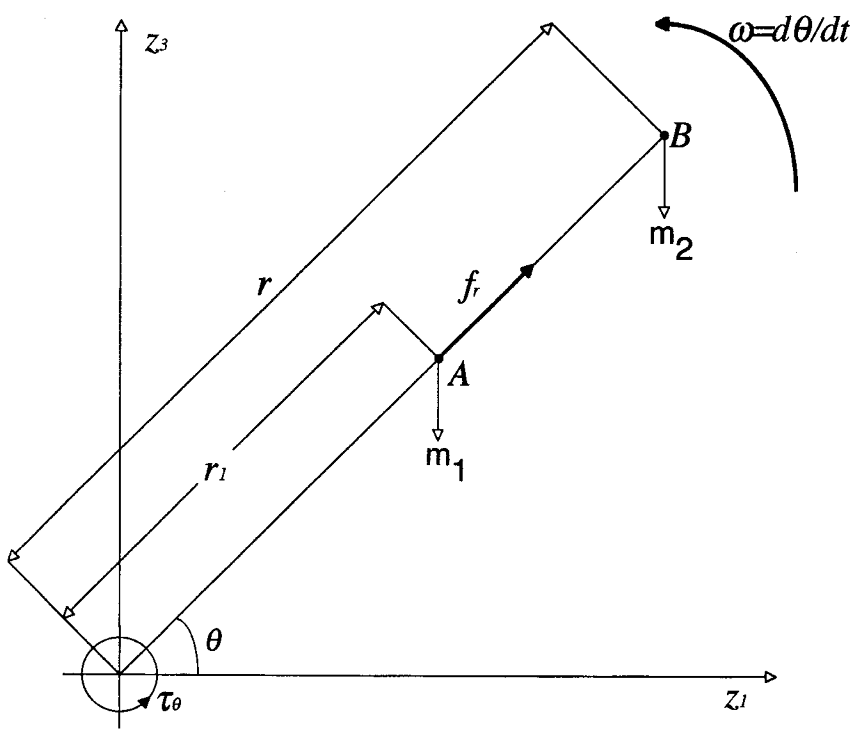

Fig. 8. A schematic representation of the planar manipulator.

$x_2 =$ the angular speed, $\dot{\theta}(t)$, of the arm
$x_3 =$ the translational position, $r(t)$, of the mass m_2
$x_4 =$ the translational speed, $\dot{r}(t)$, of the mass m_2
$u_1 =$ the actuator torque, $\tau_\theta(t)$
$u_2 =$ the linear torque, $f_r(t)$

These yield

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = (u_1 - 2m_2x_2x_3x_4)/(m_1r_1^2 + m_2x_3^2)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = (u_2 + m_2x_2^2x_3)/m_2$$

If we position the base of the robot arm at the point (b_1c_1, b_1c_2) in the horizontal z_1–z_3 plane, then the point

$$(x_3 \cos x_1 + b_1c_1, x_3 \sin x_1 + b_1c_2)$$

represents the position of the gripper at time t. We shall refer to this point as 'Robot 1'.

For the second robot arm, if we assume that

$y_1 =$ the angular position, $\theta_2(t)$, of the arm
$y_2 =$ the angular speed, $\dot{\theta}_2(t)$, of the arm
$y_3 =$ the translational position, $s(t)$, of the mass m_4
$y_4 =$ the translational speed, $\dot{s}(t)$, of the mass m_4
$v_1 =$ the actuator torque, $\tau_{\theta_2}(t)$
$v_2 =$ the linear torque, $f_s(t)$

then we will have a similar system of differential equations which will govern the arm's motion in the z_1–z_3 plane. If we let (b_2c_1, b_2c_2) be the base of the arm, then the point

$$(y_3 \cos y_1 + b_2c_1, y_3 \sin y_1 + b_2c_2)$$

represents the position of the arm's gripper at time t. We shall refer to this point as 'Robot 2'.

We have thus the state-space equations

$$\left.\begin{aligned}
\dot{x}_1 &= x_2, & \dot{y}_1 &= y_2 \\
\dot{x}_2 &= \frac{u_1 - 2m_2x_2x_3x_4}{m_1r_1^2 + m_2x_3^2}, & \dot{y}_2 &= \frac{v_1 - 2m_4y_2y_3y_4}{m_3r_2^2 + m_4y_3^2} \\
\dot{x}_3 &= x_4, & \dot{y}_3 &= y_4 \\
\dot{x}_4 &= \frac{u_2 + m_2x_2^2x_3}{m_2}, & \dot{y}_4 &= \frac{v_2 + m_4y_2^2y_3}{m_4}
\end{aligned}\right\} \tag{7}$$

to describe the angular and translational velocities $(\dot{x}_1, \dot{x}_3) = (x_2, x_4)$ and $(\dot{y}_1, \dot{y}_3) = (y_2, y_4)$ and the associated accelerations $(\dot{x}_2, \dot{x}_4) = (u_1, u_2)$ and $(\dot{y}_2, \dot{y}_4) = (v_1, v_2)$ of the grippers. Let us use the vector notations

$$\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4, \qquad \mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbf{R}^4$$

to refer to the angular and translational position and velocity components of Robot 1 and Robot 2, respectively.

We assume that the controllers $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ can move the grippers to their respective targets, which are defined as

$$T_1 = \{(z_1, z_3) \in \mathbf{R}^2 \colon (z_1 - p_1 c_1)^2 + (z_3 - p_1 c_2)^2 \leqslant rp_1^2\}$$

$$T_2 = \{(z_1, z_3) \in \mathbf{R}^2 \colon (z_1 - p_2 c_1)^2 + (z_3 - p_2 c_2)^2 \leqslant rp_2^2\}$$

Thus, a fixed antitarget of Robot 2 is $AT_1^2 = T_1$. Another fixed antitarget of Robot 2 is the base of Robot 1 where the first link is. The link is of fixed length r_1 and therefore the base can be given as the obstacle

$$AT_2^2 = \{(z_1, z_3) \in \mathbf{R}^2 \colon (z_1 - b_1 c_1)^2 + (z_3 - b_1 c_2)^2 \leqslant r_1^2\}$$

Now, the second link of the arm of length $x_3(t) = r(t)$ varies in length as the gripper moves about the workspace. Thus, Robot 2 will encounter its third obstacle, the moving object

$$AT_3^2 = \left\{(z_1, z_3) \in \mathbf{R}^2 \colon \left[z_1 - \left(\frac{x_3 - r_1}{2} \cos x_1 + b_1 c_1 \right) \right]^2 \right.$$
$$\left. + \left[z_3 - \left(\frac{x_3 - r_1}{2} \sin x_1 + b_1 c_2 \right) \right]^2 \leqslant \left(\frac{x_3 - r_1}{2} + \varepsilon_1 \right)^2 \right\}$$

where the 'safety parameter' $\varepsilon_1 > 0$ is necessary to protect the gripper of Robot 1.

Similarly, Robot 1 will encounter three obstacles. These are

$$AT_1^1 = T_2$$

$$AT_2^1 = \{(z_1, z_3) \in \mathbf{R}^2 \colon (z_1 - b_2 c_1)^2 + (z_3 - b_2 c_2)^2 \leqslant r_2^2\}$$

$$AT_3^1 = \left\{(z_1, z_3) \in \mathbf{R}^2 \colon \left[z_1 - \left(\frac{y_3 - r_2}{2} \cos y_1 + b_2 c_1 \right) \right]^2 \right.$$
$$\left. + \left[z_3 - \left(\frac{y_3 - r_2}{2} \sin y_1 + b_2 c_2 \right) \right]^2 \leqslant \left(\frac{y_3 - r_2}{2} + \varepsilon_2 \right)^2 \right\}$$

We can now state the control objectives as follows (see Fig. 9):

[R1] To control the movement of Robot 1 to its fixed target T_1 while ensuring it avoids the fixed antitargets $AT_1^1 = T_2$ and AT_2^1, and the moving antitarget $AT_3^1(t)$.

[R2] To control the movement of Robot 2 to its fixed target T_2 while ensuring it avoids fixed antitargets $AT_1^2 = T_1$ and AT_2^2, and the moving antitarget $AT_3^2(t)$.

4.1 System constraints as fixed antitargets

Let us assume the following:

Minimum extension of the arms

$$r_1 = r_2 = 1 \text{ (m)}$$

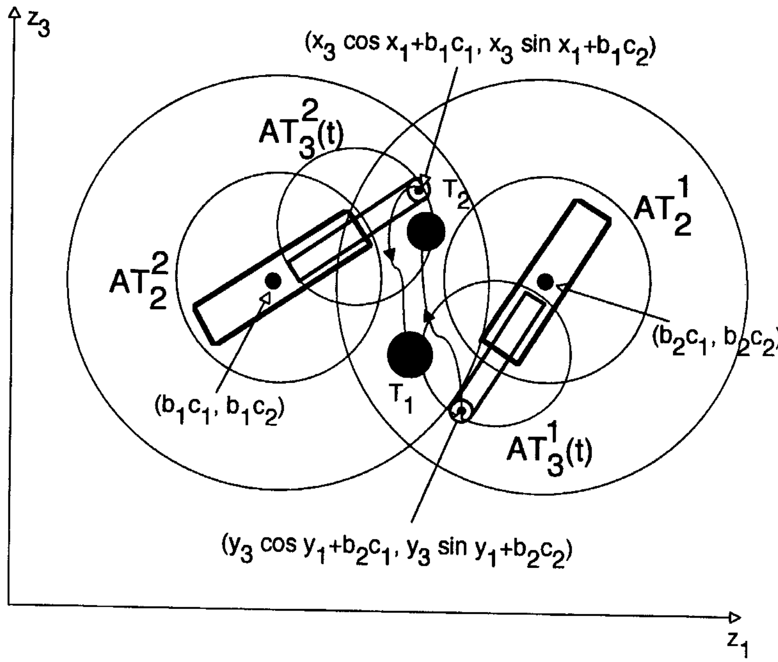

Fig. 9. Cooperation between two planar manipulators.

Maximum extension of the arms

$$x_3 = y_3 = 2 \text{ (m)}$$

Further, we assume that

$$-1 < x_2 < 1, \qquad -1 < y_2 < 1 \text{ (rad s}^{-1}\text{)}$$
$$1 < x_3 < 2, \qquad 1 < y_3 < 2 \text{ (m)}$$
$$-1 < x_4 < 1, \qquad -1 < y_4 < 1 \text{ (m s}^{-1}\text{)}$$

We now represent the above constraints as fixed antitargets.

For Robot 1, we have

$$AT_4^1 = \{\mathbf{x} \in \mathbf{R}^4 : x_2 + 1 \leqslant 0\}, \qquad AT_5^1 = \{\mathbf{x} \in \mathbf{R}^4 : x_2 - 1 \geqslant 0\}$$
$$AT_6^1 = \{\mathbf{x} \in \mathbf{R}^4 : x_3 - 1 \leqslant 0\}, \qquad AT_7^1 = \{\mathbf{x} \in \mathbf{R}^4 : x_3 - 2 \geqslant 0\}$$
$$AT_8^1 = \{\mathbf{x} \in \mathbf{R}^4 : x_4 + 1 \leqslant 0\}, \qquad AT_9^1 = \{\mathbf{x} \in \mathbf{R}^4 : x_4 - 1 \geqslant 0\}$$

and for Robot 2

$$AT_4^2 = \{\mathbf{y} \in \mathbf{R}^4 : y_2 + 1 \leqslant 0\}, \qquad AT_5^2 = \{\mathbf{y} \in \mathbf{R}^4 : y_2 - 1 \geqslant 0\}$$
$$AT_6^2 = \{\mathbf{y} \in \mathbf{R}^4 : y_3 - 1 \leqslant 0\}, \qquad AT_7^2 = \{\mathbf{y} \in \mathbf{R}^4 : y_3 - 2 \geqslant 0\}$$
$$AT_8^2 = \{\mathbf{y} \in \mathbf{R}^4 : y_4 + 1 \leqslant 0\}, \qquad AT_9^2 = \{\mathbf{y} \in \mathbf{R}^4 : y_4 - 1 \geqslant 0\}$$

4.2 Obstacle avoidance and attraction to target

In this section, we simply list the functions that dictate collision avoidance and those that dictate target attraction. Beside each function, we use the symbols '$\rightarrow$' to indicate attraction and '$\leftrightarrow$' to indicate avoidance.

4.2.1 Robot 1.

Attraction

$$V_0(\mathbf{x}) = \frac{1}{2}[(x_3 \cos x_1 + b_1 c_1 - p_1 c_1)^2 + (x_3 \sin x_1 + b_1 c_2 - p_1 c_2)^2 + x_2^2 + x_4^2] \to T_1$$

Avoidance

$$V_1(\mathbf{x}) = \frac{1}{2}[(x_3 \cos x_1 + b_1 c_1 - p_2 c_1)^2 + (x_3 \sin x_1 + b_1 c_2 - p_2 c_2)^2 - rp_2^2] \leftrightarrow AT_1^1$$

$$V_2(\mathbf{x}) = \frac{1}{2}[(x_3 \cos x_1 + b_1 c_1 - b_2 c_1)^2 + (x_3 \sin x_1 + b_1 c_2 - b_2 c_2)^2 - r_2^2] \leftrightarrow AT_2^1$$

$$V_3(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\left\{\left[x_3 \cos x_1 + b_1 c_1 - \left(\frac{y_3 - r_2}{2}\cos y_1 + b_2 c_1\right)\right]^2 + \left[x_3 \sin x_1 + b_1 c_2 - \left(\frac{y_3 - r_2}{2}\sin y_1 + b_2 c_2\right)\right]^2 - \left(\frac{y_3 - r_2}{2} + \varepsilon_2\right)^2\right\} \leftrightarrow AT_3^1$$

$$V_4(\mathbf{x}) = \frac{1}{2}(x_2 + 1)(1 - x_2) \leftrightarrow AT_4^1 \text{ and } AT_5^1$$

$$V_5(\mathbf{x}) = \frac{1}{2}(x_3 - 1)(2 - x_3) \leftrightarrow AT_6^1 \text{ and } AT_7^1$$

$$V_6(\mathbf{x}) = \frac{1}{2}(x_4 + 1)(1 - x_4) \leftrightarrow AT_8^1 \text{ and } AT_9^1$$

To ensure that the Liapunov function to be proposed is zero at the center of the targets, we use the function, for Robot 1

$$F(\mathbf{x}) = \frac{1}{2}[(x_3 \cos x_1 + b_1 c_1 - p_1 c_1)^2 + (x_3 \sin x_1 + b_1 c_2 - p_1 c_2)^2]$$

Introduce control parameters $\beta_{1k} > 0$, $k = 1, \ldots, 6$. Then for attraction and collision avoidance, we consider

$$V(\mathbf{x}, \mathbf{y}) = V_0 + F \sum_{k=1}^{6} \frac{\beta_{1k}}{V_k}$$

which is defined, continuous and positive on the domain

$$D(V) = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^4 \times \mathbf{R}^4 \colon V_k > 0, k = 1, \ldots, 6\}$$

4.2.2 Robot 2. Similarly, for Robot 2, we consider the functions

$$W_0(\mathbf{y}) = \frac{1}{2}[(y_3 \cos y_1 + b_2 c_1 - p_2 c_1)^2 + (y_3 \sin y_1 + b_2 c_2 - p_2 c_2)^2 + y_2^2 + y_4^2] \to T_2$$

$$W_1(\mathbf{y}) = \frac{1}{2}[(y_3 \cos y_1 + b_2 c_1 - p_1 c_1)^2 + (y_3 \sin y_1 + b_2 c_2 - p_1 c_2)^2 - rp_1^2] \leftrightarrow AT_1^2$$

$$W_2(\mathbf{y}) = \frac{1}{2}[(y_3\cos y_1 + b_2c_1 - b_1c_1)^2 + (y_3\sin y_1 + b_2c_2 - b_1c_2)^2 - r_1^2] \leftrightarrow AT_2^2$$

$$W_3(\mathbf{x},\mathbf{y}) = \frac{1}{2}\left\{\left[y_3\cos y_1 + b_2c_1 - \left(\frac{x_3-r_1}{2}\cos x_1 + b_1c_1\right)\right]^2 + \left[y_3\sin y_1 + b_2c_2 - \left(\frac{x_3-r_1}{2}\sin x_1 + b_1c_2\right)\right]^2 - \left(\frac{x_3-r_1}{2}+\varepsilon_1\right)^2\right\} \leftrightarrow AT_3^2$$

$$W_4(\mathbf{y}) = \frac{1}{2}(y_2+1)(1-y_2) \leftrightarrow AT_4^2 \text{ and } AT_5^2$$

$$W_5(\mathbf{y}) = \frac{1}{2}(y_3-1)(2-y_3) \leftrightarrow AT_6^2 \text{ and } AT_7^2$$

$$W_6(\mathbf{y}) = \frac{1}{2}(y_4+1)(1-y_4) \leftrightarrow AT_8^2 \text{ and } AT_9^2$$

For the Liapunov function to be zero at the center of the targets, we use, for Robot 2

$$G(\mathbf{y}) = \frac{1}{2}[(y_3\cos y_1 + b_2c_1 - p_2c_1)^2 + (y_3\sin y_1 + b_2c_2 - p_2c_2)^2]$$

Then, for a collision-free path, we consider

$$W(\mathbf{x},\mathbf{y}) = W_0 + G\sum_{k=1}^{6}\frac{\beta_{2k}}{W_k}, \qquad \beta_{2k} > 0, k = 1,\ldots,6$$

This function is defined, continuous and positive on the domain

$$D(W) = \{(\mathbf{x},\mathbf{y}) \in \mathbf{R}^4 \times \mathbf{R}^4 \colon W_k > 0, k = 1,\ldots,6\}$$

4.3 Liapunov function

Introduce as a tentative Liapunov function for system (7)

$$\begin{aligned} L(\mathbf{x},\mathbf{y}) &= V(\mathbf{x},\mathbf{y}) + W(\mathbf{x},\mathbf{y}) \\ &= V_0 + F\sum_{k=1}^{6}\frac{\beta_{1k}}{V_k} + W_0 + G\sum_{k=1}^{6}\frac{\beta_{2k}}{W_k} \end{aligned}$$

which is defined, continuous and positive on the open domain $D(V)\cap D(W)$. Along a solution of system (7), we have

$$\dot{L}_{(7)} = \dot{V}_0 + \sum_{k=1}^{6}\beta_{1k}\left(\frac{V_k\dot{F} - F\dot{V}_k}{V_k^2}\right) + \dot{W}_0 + \sum_{k=1}^{6}\beta_{2k}\left(\frac{W_k\dot{G} - G\dot{W}_k}{W_k^2}\right)$$

Treating the velocity components separately, we have

$$\dot{L}_{(7)} = \left[\left(\frac{V_4^2 + \beta_{14}F}{V_4^2}\right)\left(\frac{u_1 - 2m_2x_2x_3x_4}{m_1r_1^2 + m_2x_3^2}\right) + f_2(\mathbf{x},\mathbf{y})\right]x_2$$

$$+\left[\left(\frac{V_6^2+\beta_{16}F}{V_6^2}\right)\left(\frac{u_2+m_2x_2^2x_3}{m_2}\right)+f_4(\mathbf{x},\mathbf{y})\right]x_4$$

$$+\left[\left(\frac{W_4^2+\beta_{24}G}{W_4^2}\right)\left(\frac{v_1-2m_4y_2y_3y_4}{m_3r_2^2+m_4y_3^2}\right)+g_2(\mathbf{x},\mathbf{y})\right]y_2$$

$$+\left[\left(\frac{W_6^2+\beta_{26}G}{W_6^2}\right)\left(\frac{v_2+m_4y_2^2y_3}{m_4}\right)+g_4(\mathbf{x},\mathbf{y})\right]y_4$$

where

$$\begin{aligned}
f_2 = x_3 (&[\cos x_1 (x_3 \sin x_1 + b_1c_2 - p_1c_2) \\
&- \sin x_1 (x_3 \cos x_1 + b_1c_1 - p_1c_1)] \times \left(1+\sum_{k=1}^{6}\frac{\beta_{1k}}{V_k}\right) \\
&- [\cos x_1 (x_3 \sin x_1 + b_1c_2 - p_2c_2) - \sin x_1 (x_3 \cos x_1 + b_1c_1 - p_2c_1)] \times \frac{\beta_{11}F}{V_1^2} \\
&- [\cos x_1 (x_3 \sin x_1 + b_1c_2 - b_2c_2) - \sin x_1 (x_3 \cos x_1 + b_1c_1 - b_2c_1)] \times \frac{\beta_{12}F}{V_2^2} \\
&- \left\{\cos x_1\left[x_3 \sin x_1 + b_1c_2 - \left(\frac{y_3-r_2}{2}\sin y_1 + b_2c_2\right)\right]\right. \\
&\left.- \sin x_1\left[x_3 \cos x_1 + b_1c_1 - \left(\frac{y_3-r_2}{2}\cos y_1 + b_2c_1\right)\right]\right\} \times \frac{\beta_{13}F}{V_3^2} \\
&+ (x_3 - r_1)\left\{\cos x_1\left[y_3 \sin y_1 + b_2c_2 - \left(\frac{x_3-r_1}{2}\sin x_1 + b_1c_2\right)\right]\Big/2\right. \\
&\left.- \sin x_1\left[y_3 \cos y_1 + b_2c_1 - \left(\frac{x_3-r_1}{2}\cos x_1 + b_1c_1\right)\right]\Big/2\right\} \times \frac{\beta_{23}G}{W_3^2}
\end{aligned}$$

$$\begin{aligned}
f_4 = &[\cos x_1 (x_3 \cos x_1 + b_1c_1 - p_1c_1) \\
&+ \sin x_1 (x_3 \sin x_1 + b_1c_2 - p_1c_2)] \times \left(1+\sum_{k=1}^{6}\frac{\beta_{1k}}{V_k}\right) \\
&- [\cos x_1 (x_3 \cos x_1 + b_1c_1 - p_2c_1) + \sin x_1 (x_3 \sin x_1 + b_1c_2 - p_2c_2)] \times \frac{\beta_{11}F}{V_1^2} \\
&- [\cos x_1 (x_3 \cos x_1 + b_1c_1 - b_2c_1) + \sin x_1 (x_3 \sin x_1 + b_1c_2 - b_2c_2)] \times \frac{\beta_{12}F}{V_2^2} \\
&- \left\{\cos x_1\left[x_3 \cos x_1 + b_1c_1 - \left(\frac{y_3-r_2}{2}\cos y_1 + b_2c_1\right)\right]\right.
\end{aligned}$$

$$+\sin x_1\left[x_3\sin x_1+b_1c_2-\left(\frac{y_3-r_2}{2}\sin y_1+b_2c_2\right)\right]\Bigg\}\times\frac{\beta_{13}F}{V_3^2}-\left(\frac{3}{2}-x_3\right)\times\frac{\beta_{15}F}{V_5^2}$$

$$+\left\{\cos x_1\left[y_3\cos y_1+b_2c_1-\left(\frac{x_3-r_1}{2}\cos x_1+b_1c_1\right)\right]\Big/2\right.$$

$$+\sin x_1\left[y_3\sin y_1+b_2c_2-\left(\frac{x_3-r_1}{2}\sin x_1+b_1c_2\right)\right]\Big/2$$

$$\left.+\left(\frac{x_3-r_1}{2}+\varepsilon_1\right)\Big/2\right\}\times\frac{\beta_{23}G}{W_3^2}$$

$$g_2=y_3([\cos y_1(y_3\sin y_1+b_2c_2-p_2c_2)$$

$$-\sin y_1(y_3\cos y_1+b_2c_1-p_2c_1)]\times\left(1+\sum_{k=1}^{6}\frac{\beta_{2k}}{W_k}\right)$$

$$-[\cos y_1(y_3\sin y_1+b_2c_2-p_1c_2)-\sin y_1(y_3\cos y_1+b_2c_1-p_1c_1)]\times\frac{\beta_{21}G}{W_1^2}$$

$$-[\cos y_1(y_3\sin y_1+b_2c_2-b_1c_2)-\sin y_1(y_3\cos y_1+b_2c_1-b_1c_1)]\times\frac{\beta_{22}G}{W_2^2}$$

$$-\left\{\cos y_1\left[y_3\sin y_1+b_2c_2-\left(\frac{x_3-r_1}{2}\sin x_1+b_1c_2\right)\right]\right.$$

$$\left.-\sin y_1\left[y_3\cos y_1+b_2c_1-\left(\frac{x_3-r_1}{2}\cos x_1+b_1c_1\right)\right]\right\}\times\frac{\beta_{23}G}{W_3^2}$$

$$+(y_3-r_2)\left\{\cos y_1\left[x_3\sin x_1+b_1c_2-\left(\frac{y_3-r_2}{2}\sin y_1+b_2c_2\right)\right]\Big/2\right.$$

$$\left.-\sin y_1\left[x_3\cos x_1+b_1c_1-\left(\frac{y_3-r_1}{2}\cos y_1+b_2c_1\right)\right]\Big/2\right\}\times\frac{\beta_{13}F}{V_3^2}$$

$$g_4=\cos y_1(y_3\cos y_1+b_2c_1-p_2c_1)$$

$$+\sin y_1(y_3\sin y_1+b_2c_2-p_2c_2)]\times\left(1+\sum_{k=1}^{6}\frac{\beta_{2k}}{W_k}\right)$$

$$-[\cos y_1(y_3\cos y_1+b_2c_1-p_1c_1)+\sin y_1(y_3\sin y_1+b_2c_2-p_1c_2)]\times\frac{\beta_{21}G}{W_1^2}$$

$$-[\cos y_1(y_3\cos y_1+b_2c_1-b_1c_1)+\sin y_1(y_3\sin y_1+b_2c_2-b_1c_2)]\times\frac{\beta_{22}G}{W_2^2}$$

$$-\left\{\cos y_1\left[y_3\cos y_1+b_2c_1-\left(\frac{x_3-r_1}{2}\cos x_1+b_1c_1\right)\right]\right.$$

$$+\sin y_1\left[y_3\sin y_1+b_2c_2-\left(\frac{x_3-r_1}{2}\sin x_1+b_1c_2\right)\right]\Bigg\}\times\frac{\beta_{23}G}{W_3^2}-\left(\frac{3}{2}-y_3\right)\times\frac{\beta_{25}G}{W_5^2}$$

$$+\left\{\cos y_1\left[x_3\cos x_1+b_1c_1-\left(\frac{y_3-r_2}{2}\cos y_1+b_2c_1\right)\right]\Big/2\right.$$

$$+\sin y_1\left[x_3\sin x_1+b_1c_2-\left(\frac{y_3-r_2}{2}\sin y_1+b_2c_2\right)\right]\Big/2$$

$$\left.+\left(\frac{y_3-r_2}{2}+\varepsilon_2\right)\Big/2\right\}\times\frac{\beta_{13}F}{V_3^2}$$

Then using the convergence parameters $\rho_1\gamma_1>0$, $\rho_1\gamma_2>0$, $\rho_2\gamma_1>0$, $\rho_2\gamma_2>0$, such that

$$-\rho_1\gamma_2x_2=\left(\frac{V_4^2+\beta_{14}F}{V_4^2}\right)\left(\frac{u_1-2m_2x_2x_3x_4}{m_1r_1^2+m_2x_3^2}\right)+f_2(\mathbf{x},\mathbf{y})$$

$$-\rho_1\gamma_4x_4=\left(\frac{V_6^2+\beta_{16}F}{V_6^2}\right)\left(\frac{u_2+m_2x_2^2x_3}{m_2}\right)+f_4(\mathbf{x},\mathbf{y})$$

$$-\rho_2\gamma_2y_2=\left(\frac{W_4^2+\beta_{24}G}{W_4^2}\right)\left(\frac{v_1-2m_4y_2y_3y_4}{m_3r_2^2+m_4y_3^2}\right)+g_2(\mathbf{x},\mathbf{y})$$

$$-\rho_2\gamma_4y_4=\left(\frac{W_6^2+\beta_{26}G}{W_6^2}\right)\left(\frac{v_2+m_4y_2^2y_3}{m_4}\right)+g_4(\mathbf{x},\mathbf{y})$$

we have

$$\dot{L}_{(7)}=-\rho_1\gamma_2x_2^2-\rho_1\gamma_4x_4^2-\rho_2\gamma_2y_2^2-\rho_2\gamma_4y_4^2$$

and

$$u_1(\mathbf{x},\mathbf{y})=\frac{(m_1r_1^2+m_2x_3^2)[-\rho_1\gamma_2x_2-f_2(\mathbf{x},\mathbf{y})]V_4^2}{V_4^2+\beta_{14}F}+2m_2x_2x_3x_4 \tag{8}$$

$$u(\mathbf{x},\mathbf{y})=\frac{m_2[-\rho_1\gamma_4x_4-f_4(\mathbf{x},\mathbf{y})]V_6^2}{V_6^2+\beta_{16}F}-m_2x_2^2x_3 \tag{9}$$

$$v_1(\mathbf{x},\mathbf{y})=\frac{(m_3r_2^2+m_4y_3^2)[-\rho_2\gamma_2y_2-g_2(\mathbf{x},\mathbf{y})]W_4^2}{W_4^2+\beta_{24}G}+2m_4y_2y_3y_4 \tag{10}$$

$$v_2(\mathbf{x},\mathbf{y})=\frac{m_4[-\rho_2\gamma_4y_{4\beta}-g_4(\mathbf{x},\mathbf{y})]W_6^2}{W_6^2+\beta_{26}G}-m_4y_2^2y_3 \tag{11}$$

the nonlinear analytic forms of control laws for the planar movement of Robot 1 and Robot 2.

Let us restrict the linear accelerations $\ddot{r}(t)$ and $\ddot{s}(t)$, and angular accelerations $\ddot{\theta}(t)$ and $\ddot{\theta}_2(t)$, as follows:

$$\max|\dot{x}_2| = 1, \qquad \max|\dot{x}_4| = 1$$

$$\max|\dot{y}_2| = 1, \qquad \max|\dot{y}_4| = 1$$

and let

$$(\tilde{u}_1, \tilde{u}_2) = (u_1, u_2), \qquad (\tilde{v}_1, \tilde{v}_2) = (v_1, v_2)$$

Then our trajectory control scheme becomes, for Robot 1

If $\dot{x}_2 > 1.0$ then $u_1 = (m_1 r_1^2 + m_2 x_3^2) + 2m_2 x_3 x_4 x_2$
If $\dot{x}_2 < -1.0$ then $u_1 = -(m_1 r_1^2 + m_2 x_3^2) + 2m_2 x_3 x_4 x_2$
Else $u_1 = \tilde{u}_1$
If $\dot{x}_4 > 1.0$ then $u_2 = m_2 - m_2 x_3 x_2^2$
If $\dot{x}_4 < -1.0$ then $u_2 = -m_2 - m_2 x_3 x_2^2$
Else $u_2 = \tilde{u}_2$

and a similar one for Robot 2.

Finally, we know that in the z_1–z_3 plane, the centers of the targets are $(z_1, z_3) = (p_1c_1, p_1c_2)$ and $(z_1, z_3) = (p_2c_1, p_2c_2)$ for T_1 and T_2, respectively. When the grippers reach the centers of these targets, the length of the arm of Robot 1 becomes $\sqrt{(p_1c_1 - b_1c_1)^2 + (p_1c_2 - b_1c_2 - b_1c_2)^2}$ and the length of the arm of Robot 2 becomes $\sqrt{(p_2c_1 - b_2c_1)^2 + (p_2c_2 - b_2c_2)^2}$. Thus, solving for x_1 and y_1 in the equations

$$z_1 = x_3 \cos x_1 + b_1c_1 = p_1c_1, \qquad z_3 = x_3 \sin x_1 + b_1c_2 = p_1c_2$$

and

$$z_1 = y_3 \cos y_1 + b_2c_1 = p_2c_1, \qquad z_3 = y_3 \sin y_1 + b_2c_2 = p_2c_2$$

respectively, and letting

$$\mathbf{x}_e = \left(\tan^{-1}\left(\frac{p_1c_2 - b_1c_2}{p_1c_1 - b_1c_1}\right) + n\pi, 0, \sqrt{(p_1c_1 - b_1c_1)^2 + (p_1c_2 - b_1c_2)^2}, 0\right) \in \mathbf{R}^4$$

where $p_1c_1 \neq b_1c_1$ and $n = 0$ or $n = 1$ or $n = 2$, depending on the sign of $(p_1c_1 - b_1c_1)$ and $(p_1c_2 - b_1c_2)$, and

$$\mathbf{y}_e = \left(\tan^{-1}\left(\frac{p_2c_2 - b_2c_2}{p_2c_1 - b_2c_1}\right) + n\pi, 0, \sqrt{(p_2c_1 - b_2c_1)^2 + (p_2c_2 - b_2c_2)^2}, 0\right) \in \mathbf{R}^4$$

where $p_2c_1 \neq b_2c_1$ and $n = 0$ or $n = 1$ or 2, depending on the sign of $(p_2c_1 - b_2c_1)$ and $(p_2c_2 - b_2c_2)$, we easily have

$$(\mathbf{x}, \mathbf{y}) = (\mathbf{x}_e, \mathbf{y}_e) \in D(V) \cap D(W)$$

as an equilibrium state of system (7), the state that represents the centers of the targets where the kinetic energy is zero.

We can thus declare the properties of L as follows:

(i) $L(\mathbf{x}, \mathbf{y})$ is continuous and has first partial derivatives in the region $D(V) \cap D(W)$ in the neighborhood of the stable equilibrium state $(\mathbf{x}_e, \mathbf{y}_e)$,
(ii) $L(\mathbf{x}_e, \mathbf{y}_e) = 0$,
(iii) $L(\mathbf{x}, \mathbf{y}) > 0 \;\forall\, (\mathbf{x}, \mathbf{y}) \in D(V) \cap D(W) \backslash (\mathbf{x}_e, \mathbf{y}_e)$,
(iv) $\dot{L}_{(7)}(\mathbf{x}, \mathbf{y}) \leqslant 0 \;\forall\, (\mathbf{x}, \mathbf{y}) \in D(V) \cap D(W)$.

Hence, L is a Liapunov function for system (7).

The following theorem summarizes the above discussions:

Theorem 2. The equilibrium state $(\mathbf{x}_e, \mathbf{y}_e)$ of system (7) is stable provided u_1, u_2, v_1, and v_2 are defined as (8), (9), (10), and (11), respectively.

4.4 Simulations

Example 3. This example simply illustrates the collision-avoidance capabilities of the robot arms, with no effort made to obtain good trajectories in terms of smoothness and time. Table 3 gives the details and Fig. 10 shows the collision-free paths. Both robots reach their targets in 29.4 units of time.

Table 3. Example 3

Time interval, RK4 step size	$[0, 30]$, 0.01
Masses	$m_1 = m_3 = 7$ kg, $m_2 = m_4 = 1$ kg
Base centers	$(b_1c_1, b_1c_2) = (-1.5, 0.0)$
	$(b_2c_1, b_2c_2) = (1.5, 0.0)$
Targets centers	$(p_1c_1, p_1c_2) = (0.0, 0.6)$
	$(p_2c_1, p_2c_2) = (0.0, -0.6)$
Targets radii	$rp_1 = rp_2 = 0.1$ m
Initial states	$\mathbf{x}_0 = (5\pi/4, 0.1, 1.8, 0.1)$
	$\mathbf{y}_0 = (\pi/4, 0.1, 1.8, 0.1)$
Safety parameters	$\varepsilon_1 = \varepsilon_2 = 0.01$ m
Control parameters	$\beta_{11} = \beta_{13} = \beta_{14} = \beta_{15} = \beta_{16} = 1.0$, $\beta_{12} = 0.001$
	$\beta_{21} = \beta_{23} = \beta_{24} = \beta_{25} = \beta_{26} = 1.0$, $\beta_{22} = 0.001$
Convergence parameters	$\rho_1\gamma_2 = \rho_1\gamma_4 = 60.0$
	$\rho_2\gamma_2 = \rho_2\gamma_4 = 60.0$

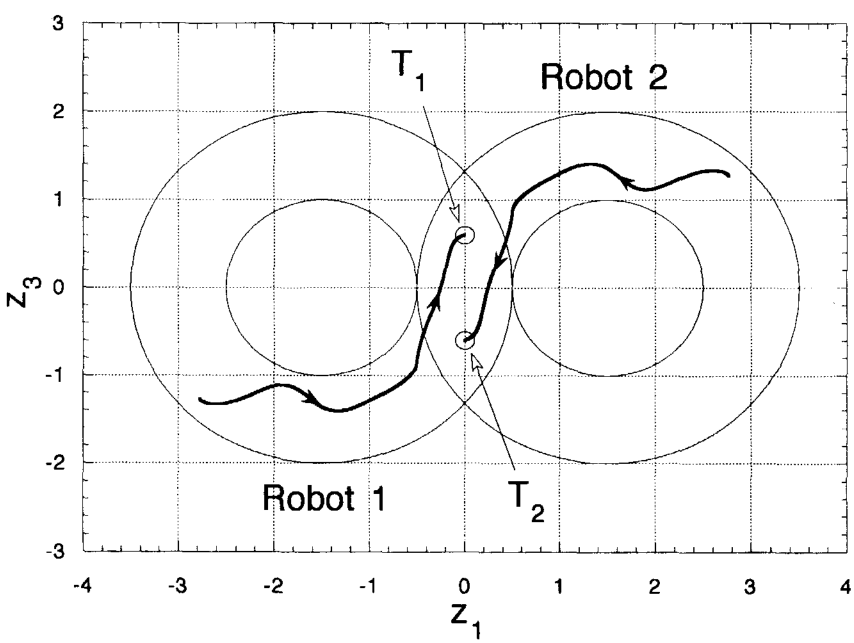

Fig. 10. Collision avoidance between the planar manipulators (Example 3).

Table 4. Example 4

Time interval, RK4 step size	$[0, 50]$, 0.017
Masses	$m_1 = m_3 = 7$ kg, $m_2 = m_4 = 1$ kg
Base centers	$(b_1c_1, b_1c_2) = (-1.0, 0.0)$
	$(b_2c_1, b_2c_2) = (1.0, 0.0)$
Targets centers	$(p_2c_1, p_1c_2) = (0.0, 0.6)$
	$(p_2c_1, p_2c_2) = (0.0, -0.6)$
Targets radii	$rp_1 = rp_2 = 0.1$ m
Initial states	$\mathbf{x}_0 = (4.0, -0.2, 1.5, -0.1)$
	$\mathbf{y}_0 = (2.0, 0.1, 1.3, 0.1)$
Safety parameters	$\varepsilon_1 = \varepsilon_2 = 0.01$ m
Control parameters	$\beta_{11} = \beta_{12} = \beta_{13} = 5.0$, $\beta_{14} = \beta_{16} = 0.05$, $\beta_{15} = 1.0$
	$\beta_{12} = \beta_{22} = \beta_{23} = 2.0$, $\beta_{24} = \beta_{26} = 0.01$, $\beta_{25} = 0.05$
Convergence parameters	$\rho_1\gamma_2 = \rho_1\gamma_4 = 40.0$
	$\rho_2\gamma_2 = \rho_2\gamma_4 = 30.0$

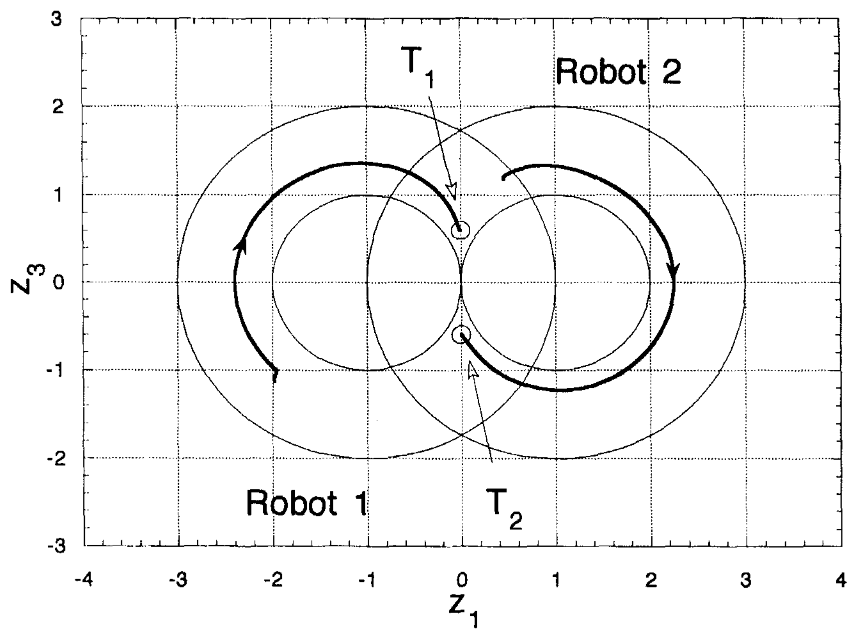

Fig. 11. For each manipulator, the shortest path leading to the target is blocked by the base of the other manipulator (Example 4).

Example 4. A difficult situation arises when the bases of the robots prevent the shortest access to the targets. However, with appropriate control parameters (Table 4), chosen along the lines stated in Section 3.5, it is easy to redirect the robots (Fig. 11). In this example, Robot 1 reaches its target in 36.3 units of time, and Robot 2 does so in 46.9 units of time.

5 Conclusion

The direct method of Liapunov is a promising tool in solving the findpath problem. In this paper, we have shown how the method could be applied to the two-dimensional case. An application involving the cooperation of two planar robot arms was considered. The method is easy to apply and offers a viable alternative to the available theoretical methods.

The next major issues are the search for a Liapunov function that guarantees at least asymptotic stability and hence solve the problem of local minima traps that plagues many of the physical analogy-based algorithms, and whether the method could be extended to the more realistic three dimensions.

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References

Boyce, W. E. and DiPrima, R. C. (1992) *Elementary Differential Equations and Boundary Value Problems*, 5th edn (John Wiley, New York).

Brooks, R. A. (1983) Solving the find-path problem by good representation of free space. *IEEE Transactions on Systems, Man, and Cybernetics* **13**, 190–197.

Chien, Y. P., Koivo, A. J. and Lee, B. H. (1988) On-line generation of collision free trajectories for multiple robots. *Proceedings 1988 IEEE International Conference on Robotics and Automation*, pp. 209–214.

Connolly, C. I., Burns, J. B. and Weiss, R. (1990) Path planning using Laplace's equation. *Proceedings 1990 IEEE International Conference on Robotics and Automation*, pp. 2102–2106.

Freund, E. and Hoyer, H. (1988) Real-time pathfinding in multirobot systems including obstacle avoidance. *International Journal of Robotics Research* **7**, 42–70.

Günther, K. S. and Azarm, K. (1993) Physical analogies for rapid robot motion planning and execution in dynamic environments. *Transactions of the IMACS/SICE International Symposium on Robotics, Mechatronics and Manufacturing Systems* (North-Holland Elsevier, Amsterdam), pp. 3–8.

Herman, M. (1986) Fast, three-dimensional collision-free motion planning. *Proceedings 1986 IEEE International Conference on Robotics and Automation*, pp. 1056–1063.

Khatib, O. (1986) Real-time obstacle avoidance for manipulators and mobile robots. *International Journal of Robotics Research* **5**, 90–98.

Kim, J. and Khosla, P. (1991) Real-time obstacle avoidance using harmonic potential functions. *Proceedings 1991 IEEE International Conference on Robotics and Automation*, pp. 790–796.

LaSalle, J. and Lefschetz, S. (1961) *Stability By Liapunov's Direct Method with Applications* (Academic Press, New York).

Lozano-Pérez, T. (1983) Spatial planning: a configuration space approach. *IEEE Transactions on Computers* **32**, 108–120.

Lyapunov, A. M. (1892) The general problem of the stability of motion. *International Journal of Control: Lyapunov Centenary Issue* **55** (1992).

Schwartz, J. T. and Sharir, M. (1983) On the "Piano Movers", Problem I, The case of a two-dimensional rigid polygonal body moving amidst polygonal barriers. *Communications on Pure and Applied Mathematics* **XXXVI**, 345–398.

Shamos, M. I. and Hoey, D. (1975) Closest-point problems. *Proceedings 16th Annual FOCS Symposium*, pp. 151–162.

Sheu, P. C.-Y. and Xue, Q. (1993) *Intelligent Robotic Planning Systems* (World Scientific, Singapore).

Skowronski, J. M. (1990) *Nonlinear Liapunov Dynamics* (World Scientific, Singapore).

Stonier, R. J. (1990) Use of Liapunov techniques for collision-avoidance of robot arms. *Control and Dynamic Systems*, Vol. 35, *Advances in Control Mechanics*, Part 2 of 2 (Academic Press, New York), pp. 185–214.

Stonier, R. J. (1992) Collision-avoidance of two PR manipulators. *Lecture Notes in Control and Information Science* **170**, 248–264.

Tarassenko, L. and Blake, A. (1991) Analogue computation of collision-free paths. *Proceedings 1991 IEEE International Conference on Robotics and Automation*, pp. 540–545.

Vanualailai, J., Nakagiri, S. and Ha, J. H. (1995) Collision avoidance in a two-point system via Liapunov's second method. *Mathematics and Computers in Simulation* **35**, 125–141.