ON SEMIHEREDITARY MAXIMAL ORDERS

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Abstract

Let A be an order integral over a valuation ring V in a central simple F-algebra, where F is the fraction field of V. We show that (a) if (V^h, F^h) is the Henselization of (V, F), then A is a semihereditary maximal order if and only if $A \otimes_V V^h$ is a semihereditary maximal order, generalizing the result by Haile, Morandi and Wadsworth, and (b) if $J(V)$ is a principal ideal of V, then a semihereditary V-order is an intersection of finitely many conjugate semihereditary maximal orders; if not, then there is only one maximal order containing the V-order.

In this paper, all rings are associative with a unit element. If A is a ring, then $J(A)$ will denote its Jacobson radical, and the residue ring $A/J(A)$ will be denoted by $\overline{A}$. A ring A is called *semihereditary* if every finitely generated left (respectively right) ideal of A is projective as a left (respectively right) A-module. Let V be a commutative domain with quotient field F, and let Q be a finite-dimensional F-algebra. A subring R of Q is said to be an *order* in Q if $RF = Q$. If $V \subseteq Z(R)$, then R is said to be a V-*order* if, in addition, R is integral over V. If R is maximal with respect to inclusion among V-orders of Q, then R is called a *maximal* V-order (or just a maximal order if the context is clear). Following [**9**, Chapter 9], a V-order A is called *extremal* if for any V-order B with $A \subseteq B \subseteq Q$ and $J(A) \subseteq J(B)$ we have $A = B$. If L is an additive subgroup of Q, then the subring $O_l(L) = \{x \in Q \mid xL \subseteq L\}$ of Q is called the *left multiplier* of the group L.

In this paper, Q will denote a central simple F-algebra, and V will denote a valuation ring of F of *arbitrary* Krull dimension (*rank*), unless stated otherwise. Since V is integrally closed in this case, if R is a V-order, then $Z(R) = V$. If p is a prime ideal of V, then we shall denote the localization of V at p by V_p. The Henselization of (V, F) will be denoted by (V^h, F^h) (see [**2**] for definition). If $\mathrm{rank}(V) = 1$, then the completion of (V, F) with respect to the metric induced by the valuation will be denoted by $(\hat{V}, \hat{F})$ (see [**2**, Chapter 1, §2]). Examples of semihereditary maximal orders are Azumaya algebras over valuation rings, the classical maximal orders over a discrete valuation ring (DVR), integral Dubrovin valuation rings and, more generally, integral Bézout orders over a valuation ring. More examples of semihereditary maximal orders can be found in [**8**]. It was shown in [**3**] that when B is a Bézout V-order, its Henselization, $B \otimes_V V^h$, is a semihereditary maximal order.

One of the things that we want to do in this paper is to generalize this result (Theorem 1), in a more direct manner, to arbitrary semihereditary maximal orders which need not be Bézout. If (W, L) is an extension of (V, F), then W is a flat V-module since V is semihereditary and W is torsion-free over V. Further, since $J(V)W \subseteq J(W) \neq W$, W is actually a faithfully flat V-module. In particular, if for a V-order A we have that $A \otimes_V V^h$ is a maximal V^h-order, then A has to be a maximal order. Thus the difficulty lies in proving the converse.

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In the classical setting (when V is a DVR), semihereditary V-orders are intersections of maximal V-orders [**9**, Chapter 9]. We shall generalize this result in Theorem 2.

The case of classical orders (that is, when the centre is a Noetherian integrally closed domain) has been studied extensively in non-commutative ring theory (see, for example, [**9**]). When the centre is not necessarily Noetherian, some theorems true in the classical situation whose validity one would hope to retain in the general setting either fail to hold or must be re-proved—often with a great deal more effort.

The results in the following two lemmas will be used repeatedly throughout the paper, and thus have been put in lemma form for ease of reference.

LEMMA 1. *Let A be an extremal V-order and W a proper overring of V in F. If B is a V-order containing A, then $B \subseteq WA$.*

Proof. By [**3**, Lemma 1], $J(B) \subseteq A$. Since $WJ(V) = W$, we have

$$WB = WJ(V)B \subseteq WJ(B) \subseteq WA,$$

hence the claim.

LEMMA 2. *Let (W, L) be an extension of (V, F). Let A be a maximal V-order in Q. Let B be a W-order in $Q \otimes_F L$ containing $A \otimes_V W$, and let $b \in B$. So one can write $b = \sum a_i l_i$ with $a_i \in A$ and $l_i \in L$.*

If for each i there exists $f_i \in F$ such that $l_i - f_i \in W$, then $b \in A \otimes_V W$.

Proof. We have $b = \sum a_i f_i + \sum a_i(l_i - f_i)$. But $\sum a_i(l_i - f_i) \in A \otimes_V W$, by assumption. So $\sum a_i f_i \in B \cap Q \supseteq A$. Since A is a maximal order, we have $B \cap Q = A$, and hence $b \in A + A \otimes_V W = A \otimes_V W$.

PROPOSITION. *Let (V_1, F_1) be an extension of (V, F), where V is a valuation ring of rank 1, and V is dense in V_1. Let A be a V-order in a central simple F-algebra Q, and let $A_1 = A \otimes_V V_1$. Then*:

(a) *A is semihereditary if and only if A_1 is semihereditary*;

(b) *A is a maximal V-order in Q if and only if A_1 is a maximal V_1-order in $Q \otimes_F F_1$.*

Proof. (a) Since V is dense in V_1, we may assume that $(V, F) \subseteq (V_1, F_1) \subseteq (\hat{V}, \hat{F})$. We shall show that A is semihereditary if and only if $\hat{A} = A \otimes_V \hat{V}$ is semihereditary. Once this is done, and since $\hat{A} = A_1 \otimes_{V_1} \hat{V}$, it will follow from [**4**, Proposition 3.3] that A is semihereditary if and only if A_1 is semihereditary.

So assume that A is semihereditary. Then $A^h = A \otimes_V V^h$ is semihereditary, by [**4**, Theorem 3.4]. Let $Q^h = Q \otimes_V V^h$. Then we may assume that $Q^h = M_n(D^h)$, where D^h is an F^h-central division algebra. Let Δ be the invariant valuation ring of D^h extending V^h. Then, by [**4**, Theorem 2.4], we may write $A^h = M_n(\Delta_{ij})$, where the Δ_{ij} are Δ-submodules of D^h, $\Delta_{ii} = \Delta$, and given $0 \neq \alpha \in D^h$, either $\alpha \in \Delta_{ij}$ or $\alpha^{-1} \in \Delta_{ji}$. By [**2**, Theorems 17.18, 17.10], we may suppose that $(V^h, F^h) \subseteq (\hat{V}, \hat{F})$, and thus we can assume that $(\widehat{V^h}, \widehat{F^h}) = (\hat{V}, \hat{F})$. Since (V^h, F^h) is Henselian, $\hat{D} = D^h \otimes_{F^h} \hat{F}$ is a division algebra, and if $\hat{\Delta}$ is the invariant valuation ring of $\hat{D}$ extending $\hat{V}$, then we have $\hat{\Delta} = \Delta \otimes_{V^h} \hat{V}$. Thus $\hat{A} = A \otimes_V \hat{V} = (A \otimes_V V^h) \otimes_{V^h} \hat{V} = M_n(\hat{\Delta}_{ij})$, where $\hat{\Delta}_{ii} = \hat{\Delta}$ and $\hat{\Delta}_{ij} = \Delta_{ij} \otimes_{V^h} \hat{V}$ are $\hat{\Delta}$-submodules of $\hat{D}$. Since the value group of the valuation on $\hat{D}$ with valuation ring $\hat{\Delta}$ coincides with the value group of the valuation on D^h with valuation ring Δ, we also have the following: given $0 \neq \hat{\alpha} \in \hat{D}$, then either $\hat{\alpha} \in \hat{\Delta}_{ij}$ or $\hat{\alpha}^{-1} \in \hat{\Delta}_{ji}$. Then [**4**, Theorem 2.4] tells us that $\hat{A}$ has to be semihereditary.

Conversely, if $\hat{A}$ is semihereditary, then A is semihereditary by [**4**, Proposition 3.3].

(b) That A_1 is a maximal order whenever A is follows immediately from Lemma 2 and the fact that F is dense in F_1, where F and F_1 are regarded as metric spaces with respect to the metric induced by their respective valuations. The converse follows from faithful flatness.

REMARKS. (a) The results in the Proposition are well known when V is a DVR [**9**, Theorems 40.5, 11.5].

(b) The Proposition shows that if V has rank 1, then A is a semihereditary maximal V-order if and only if A^h is a semihereditary maximal V^h-order, if and only if $\hat{A}$ is a semihereditary maximal $\hat{V}$-order.

(c) If (W, L) is an extension of (V, F), then it is not always the case that if A is a maximal V-order then $A \otimes_V W$ is a maximal W-order, even when (W, L) is an immediate extension of (V, F). Let A be the maximal V-order constructed in [**8**, Theorem 5.7(3)], where V is a generalized discrete valuation ring, and (W, L) is a 'maximal completion' of (V, F) (see [**10**, Chapter D] for definition). If $A \otimes_V W$ is a maximal W-order, then it is semihereditary by [**8**, Corollary 5.3], and therefore A is semihereditary by [**4**, Proposition 3.3]. But A is not semihereditary, as was pointed out in the discussion before [**4**, Example 1.6], and thus $A \otimes_V W$ is not a maximal order.

(d) If now (W, L) is a 'completion' of (V, F) in the sense of Bourbaki [**1**, §5], then one can employ Lemma 2 to show that A is a maximal V-order if and only if $A \otimes_V W$ is a maximal W-order. Unfortunately, unlike in the case of maximal completions, this (W, L) need not be Henselian in general when $\operatorname{rank}(V) > 1$, and hence one cannot conclude from this that the Henselization of an arbitrary maximal order is maximal. But when the order is semihereditary, we have the following.

THEOREM 1. *Let A be a V-order where V has arbitrary rank. Then A is a semihereditary maximal order if and only if $A^h = A \otimes_V V^h$ is a semihereditary maximal order.*

Proof. That A is semihereditary if and only if A^h is semihereditary was proved in [**4**, Theorem 3.4] and, independently, in [**6**, Theorem 2.7] by Marubayashi *et al.* The case $\operatorname{rank}(V) = 1$ has already been dealt with in the Proposition. So from now on we shall assume that $\operatorname{rank}(V) > 1$. Let B be a V^h-order containing A^h, and let $b \in B$. We shall use Lemma 2 to show that $b \in A^h$, and hence A^h is maximal. We distinguish two cases.

Case 1. There is a minimal proper overring of V in F, say W. Then $V/J(W)$ is a valuation ring of $\overline{W}$ of rank 1.

Let $W' = WV^h$. Since A^h is a semihereditary V^h-order, it is extremal [**4**, Theorem 1.5]. Hence, by Lemma 1, $B \subseteq W'A^h = W'A$. Thus $b = \sum a_i \tilde{w}_i$ with $a_i \in A$ and $\tilde{w}_i \in W'$. The proof of [**7**, Theorem 2] shows that $(V^h/J(W'), \overline{W'})$ is the Henselization of $(V/J(W), \overline{W})$. Since $\operatorname{rank}(V/J(W)) = 1$, $\overline{W}$ is dense in $\overline{W'}$ as in the proof of the Proposition above. Hence, for each i, there is a $w_i \in W$ such that $\tilde{w}_i - w_i \in V^h$. It then follows from Lemma 2 that $b \in A^h$.

Case 2. There is no minimal proper overring of V in F. This happens if and only if the intersection of all the proper overrings of V in F equals V.

For each prime ideal p of V, $(V^h V_p, F^h)$ is Henselian. Let (V_p^h, F^p) be the unique Henselization of (V_p, F) contained in $(V^h V_p, F^h)$ [**2**, Theorem 17.11]. Then the fields $\{F^p\}$ are linearly ordered by inclusion, and so $L = \bigcup_{p \neq J(V)} F^p$ is a subfield of F^h containing F. We shall show that $L = F^h$.

Let V_{sep} be an extension of V^h to a separable closure F_{sep} of F^h, and let $\mathscr{D}$ be the decomposition group of V_{sep} over F, that is, $\mathscr{D} = \{\sigma \in \text{Gal}(F_{\text{sep}}/F) \mid \sigma(V_{\text{sep}}) = V_{\text{sep}}\}$. Since V has no minimal proper overring in F, neither does V_{sep} have a minimal proper overring in F_{sep}. Thus V_{sep} is the intersection of its proper overrings, and therefore $\mathscr{D}$ is the intersection of the decomposition groups of the overrings. Hence F^h, which is the fixed field of $\mathscr{D}$, is the union of the fixed fields of the decomposition groups of the proper overrings of V_{sep}, which are the Henselizations of the overrings of V. Thus $L = F^h$.

$$\begin{array}{cccc}
(V^h, F^h) & A^h = A \otimes_V V^h & \subseteq & Q^h = Q \otimes_F F^h \\
| & | & & | \\
(V^p, F^p) & A^p = A \otimes_V V^p & \subseteq & Q^p = Q \otimes_F F^p \\
| & | & & | \\
(V, F) & A & \subseteq & Q
\end{array}$$

FIG. 1

One can write $b = \sum q_i \tilde{f}_i$ with $q_i \in Q$ and $\tilde{f}_i \in F^h$. Since $F^h = \bigcup_{p \neq J(V)} F^p$ and the set $\{F^p\}$ is linearly ordered by inclusion, there exists a prime ideal p of V, $p \neq J(V)$, such that $\tilde{f}_i \in F^p$ for all i, and thus $b \in Q^p := Q \otimes_F F^p$. Let $V^p = V^h \cap F^p$, $A^p = A \otimes_V V^p$ and $B^p = B \cap Q^p$ (see Fig. 1). Since the Henselization of A^p, $A^p \otimes_{V^p} V^h = (A \otimes_V V^p) \otimes_{V^p} V^h = A^h$, is semihereditary, we conclude that A^p is a semihereditary V^p-order, hence extremal. Since $B^p \supseteq A^p$, we have $B^p \subseteq V_p^h A^p = V_p^h A$ by Lemma 1. Since $b \in B^p$, $b = \sum a_i \tilde{w}_i$ with $a_i \in A$ and $\tilde{w}_i \in V_p^h$. But (V_p^h, F^p) is an immediate extension of (V_p, F). So for each i there is a $w_i \in V_p \subseteq F$ such that $\tilde{w}_i - w_i \in J(V_p^h) \subseteq V^p \subseteq V^h$. Thus $b \in A^h$ by Lemma 2, and the proof is complete.

There exists a different proof of Theorem 1 in the special case when $J(V)$ is a principal ideal of V (see [**4**, Theorem 3.5]). By [**4**, Lemma 3.6 and Proposition 3.10], we immediately have the following.

COROLLARY 1. *Let A be a semihereditary V-order in Q with $J(V)$ a non-principal ideal of V. Then A is a maximal order if and only if $J(A) = J(V)A$.*

In [**5**, Proposition 3.4], it was shown that when $J(V)$ is a non-principal ideal of V, there exists a unique maximal order containing a given extremal V-order; when $J(V)$ is principal, by [**5**, Theorem 3.5] a semihereditary V-order inside a Bézout maximal V-order is an intersection of finitely many Bézout maximal V-orders. We shall generalize this result shortly, but first we need the following result, which is well known for maximal orders over a DVR.

LEMMA 3. *Let U be a semi-local Dedekind domain with quotient field F, and let Q be a central simple F-algebra. Then*:

(a) *any two maximal U-orders in Q are conjugate*;

(b) *a U-order R is hereditary* (*that is, all left* (*respectively right*) *ideals are projective as left* (*respectively right*) *R-modules*) *if and only if $O_l(J(R)) = R$.*

Proof. (a) Let A and B be maximal U-orders in Q. Since B is a finitely generated U-module, BA is an A-lattice in Q. Also, $A \cong BA$ as right A-modules, by [**9**, p. 181, Exercise 3], since $AU_p \cong BAU_p$ for each maximal ideal p of U by [**9**, Theorem 18.7(i)]. If this isomorphism takes $1 \in A$ to $x \in BA$, then $BA = xA$ and x is invertible since BA is a full U-lattice in Q. Then $B = xAx^{-1}$, as in the proof of [**9**, Theorem 17.3(ii)].

(b) We know that R is hereditary if and only if $R_p = RU_p$ is hereditary for every $p \in \mathrm{Spec}(U)$ [**9**, 3.24], if and only if $\hat{R}_p = R_p \otimes_{U_p} \hat{U}_p$ is hereditary for every p [**9**, 40.5], if and only if $O_l(J(\hat{R}_p)) = \hat{R}_p$ for every p [**9**, 39.11 and 39.14]. Since $J(\hat{R}_p) = J(R_p) \otimes_{U_p} \hat{U}_p$, we have $O_l(J(\hat{R}_p)) = O_l(J(R_p)) \otimes_{U_p} \hat{U}_p$, by [**9**, 5.2(ii)]. Thus R is hereditary if and only if $O_l(J(R_p)) = R_p$ for every p, by [**9**, 5.2(ii)]. The result now follows from the fact that

$$O_l(J(R)) = \bigcap_p O_l(J(R))_p = \bigcap_p O_l(J(R)_p) = \bigcap_p O_l(J(R_p)).$$

(The second equality holds since $J(R)$ is a finitely generated right ideal of R.)

THEOREM 2. *If $J(V)$ is a principal ideal of V, then an extremal V-order R is an intersection of finitely many conjugate maximal orders. If $J(V)$ is not principal, then there is only one maximal order containing R.*

Proof. Let R be an extremal V-order. Suppose that $J(V)$ is a principal ideal of V. The theorem is already known in the rank-1 case, as V is a DVR in this case. So assume $\mathrm{rank}(V) > 1$. Let $p = \bigcap_{n \geq 1} J(V)^n$. Then p is a prime ideal of V, $W = V_p$ is a minimal overring of V in F, and $\widetilde{V} = V/p$ is a DVR of $\overline{W}$. Let $T = WR$. Then $\overline{T} = \mathscr{T}_1 \oplus \cdots \oplus \mathscr{T}_n$, where each $\mathscr{T}_i$ is a finite-dimensional simple algebra with centre $\mathscr{F}_i$, say, a finite field extension of $\overline{W}$. Let $\mathscr{U}_i$ denote the integral closure of $\widetilde{V}$ in $\mathscr{F}_i$. Then $\mathscr{U}_i$ is a semi-local Dedekind domain. Set $\mathscr{U} = \mathscr{U}_1 \oplus \cdots \oplus \mathscr{U}_n$. Let A be a maximal V-order containing R. By Lemma 1, $A \subseteq T$ and hence $J(T) \subseteq A$ by [**8**, Lemma 2.5(a)]. So $J(T) = J(V)J(T) \subseteq J(V)A \subseteq J(A)$. But $J(A) \subseteq J(R)$ since R is extremal [**3**, Lemma 1]. We can thus define $\widetilde{R} = R/J(T)$. We claim that $\widetilde{R} = \mathscr{R}_1 \oplus \cdots \oplus \mathscr{R}_n$, where each $\mathscr{R}_i$ is a hereditary $\mathscr{U}_i$-order of $\mathscr{T}_i$.

$\widetilde{R}$ is an extremal $\widetilde{V}$-order in $\overline{T}$: if $\mathscr{B}$ is a $\widetilde{V}$-order in $\overline{T}$ such that $\mathscr{B} \supseteq \widetilde{R}$, then $B = \{x \in T \mid x + J(T) \in \mathscr{B}\}$ is a V-order in Q containing R [**8**, Lemma 2.5(b)]. If, in addition, $J(\mathscr{B}) \supseteq J(\widetilde{R})$, then $J(B) \supseteq J(R)$ and hence $B = R$ since R is extremal, forcing the equality $\mathscr{B} = \widetilde{R}$.

Note that $\mathscr{U}\widetilde{R}$ is a $\widetilde{V}$-order containing R. Since

$$(J(\widetilde{R})\mathscr{U})^k = J(\widetilde{R})^k \mathscr{U} \subseteq J(\widetilde{V})\widetilde{R}\mathscr{U} \subseteq J(\widetilde{R}\mathscr{U})$$

for some k, we see that $(J(\widetilde{R})\mathscr{U} + J(\mathscr{U}\widetilde{R}))/J(\mathscr{U}\widetilde{R})$ is a nilpotent ideal of the semisimple Artinian ring $\overline{\mathscr{U}\widetilde{R}}$, and thus $J(\widetilde{R})\mathscr{U} \subseteq J(\mathscr{U}\widetilde{R})$, and therefore we have $\widetilde{R} = \mathscr{U}\widetilde{R}$ since $\widetilde{R}$ is an extremal $\widetilde{V}$-order. Thus we conclude that $Z(\widetilde{R}) = \mathscr{U}$.

Now let $\{e_i\}$ be the primitive central idempotents of $\overline{T}$, with $e_i\overline{T} = \mathscr{T}_i$. Then $\{e_i\} \subseteq \mathscr{U} \subseteq \widetilde{R}$, and hence if we set $\mathscr{R}_i = e_i\widetilde{R}$, we readily see that $\widetilde{R} = \mathscr{R}_1 \oplus \cdots \oplus \mathscr{R}_n$ and each $\mathscr{R}_i$ is a $\mathscr{U}_i$-order in $\mathscr{T}_i$. Since R is an extremal V-order, $O_l(J(R)) = R$ by [**4**, Proposition 1.4]. Therefore

$$\mathscr{R}_1 \oplus \cdots \oplus \mathscr{R}_n = O_l(J(R))/J(T) = O_l(J(R)/J(T))$$
$$= O_l(J(\mathscr{R}_1) \oplus \cdots \oplus J(\mathscr{R}_n)) = O_l(J(\mathscr{R}_1)) \oplus \cdots \oplus O_l(J(\mathscr{R}_n))$$

(here, $O_l(J(\mathscr{R}_i)) = \{t \in \mathscr{T}_i \mid tJ(\mathscr{R}_i) \subseteq J(\mathscr{R}_i)\}$). Thus $O_l(J(\mathscr{R}_i)) = \mathscr{R}_i$ for each i, and therefore $\mathscr{R}_i$ is hereditary by Lemma 3(b).

From the classical theory of hereditary orders (see, for example, [**9**, Theorem 40.10]), $\mathscr{R}_i = \bigcap_j \mathscr{A}_{ij}$, where $\{\mathscr{A}_{ij} \mid 1 \leq j \leq n_i\}$ is a finite set of maximal $\mathscr{U}_i$-orders of $\mathscr{T}_i$. Any two maximal $\mathscr{U}_i$-orders are conjugate in $\mathscr{T}_i$ by Lemma 3(a). Given $j_1, j_2, \ldots, j_n$, where $1 \leq j_l \leq n_l$, $1 \leq l \leq n$, let

$$A_{j_1,\ldots,j_n} = \{x \in T \mid x + J(T) \in \mathscr{A}_{1j_1} \oplus \mathscr{A}_{2j_2} \oplus \cdots \oplus \mathscr{A}_{nj_n}\}.$$

Then $A_{j_1,\ldots,j_n}/J(T)$ is a maximal $\widetilde{V}$-order in $\overline{T}$, and hence $A_{j_1,\ldots,j_n}$ is a maximal V-order in Q by [**8**, Lemma 2.5(b)] and Lemma 1. Any two such maximal V-orders are clearly conjugate, and R is the intersection of all of them.

If $J(V)$ is not principal, then there is only one maximal order containing R, as was pointed out before Lemma 3. Clearly, such an order is an intersection of maximal orders if and only if it is already a maximal order.

Since semihereditary V-orders are extremal, and overrings of semihereditary orders in Q are also semihereditary [**8**, Lemma 4.10], we immediately have the following.

COROLLARY 2. *If $J(V)$ is a principal ideal of V, then a semihereditary V-order R is an intersection of finitely many conjugate semihereditary maximal orders. If $J(V)$ is not principal, then there is only one maximal order containing R.*

REMARK. Given two maximal V-orders in Q, say A and B, it is not always the case that they are isomorphic if V is not a DVR (see the discussion in [**8**, Example 3.6]). However, if the V-order $A \cap B$ is extremal (or if A and B contain a fixed extremal V-order), then A and B are conjugate. If $J(V)$ is principal, then this follows from arguments employed in the proof of Theorem 2; if $J(V)$ is not principal, then we actually have $A = B$.

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