CROSSED PRODUCT ORDERS OVER VALUATION RINGS

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Abstract

Let V be a commutative valuation domain of arbitrary Krull-dimension (rank), with quotient field F, and let K be a finite Galois extension of F with group G, and S the integral closure of V in K. If, in the crossed product algebra $K * G$, the 2-cocycle takes values in the group of units of S, then one can form, in a natural way, a 'crossed product order' $S * G \subseteq K * G$. In the light of recent results by H. Marubayashi and Z. Yi on the homological dimension of crossed products, this paper discusses necessary and/or sufficient valuation-theoretic conditions, on the extension K/F, for the V-order $S * G$ to be semihereditary, maximal or Azumaya over V.

In this paper, all rings are associative, with a unit element. If A is a ring, $J(A)$ will denote its Jacobson radical, and the residue ring $A/J(A)$ will be denoted by $\overline{A}$. A ring A is called *left hereditary* if every left ideal of A is projective as a left A-module, and *left semihereditary* if every finitely generated left ideal of A is projective as a left A-module. Analogous definitions hold for *right hereditary* and *right semihereditary* rings. A ring is called *hereditary* if it is both left and right hereditary, and *semihereditary* if it is both left and right semihereditary. Let V be a commutative domain with quotient field F, and let Q be a finite-dimensional F-algebra. A subring R of Q is said to be an *order* in Q if $RF = Q$. If $V \subseteq Z(R)$, then R is said to be a *V-order* if in addition R is integral over V. If R is maximal with respect to inclusion among V-orders of Q, then R is called a *maximal V-order* (or just *maximal order* if the context is clear).

In this paper, Q will denote a central simple F-algebra, and V a valuation ring of F of *arbitrary* Krull dimension, unless stated otherwise, and K/F will be a finite Galois extension with group G. Let S be the integral closure of V in K, and let $U(S)$ be its group of units. Now consider a normalised two-cocycle $f : G \times G \mapsto U(S)$; that is, a function satisfying $\sigma(f(\tau,\gamma))f(\sigma,\tau\gamma) = f(\sigma,\tau)f(\sigma\tau,\gamma)$ for all $\sigma,\tau,\gamma \in G$ and $f(1,\sigma) = f(\sigma,1) = 1$ for all $\sigma \in G$. From such a cocycle, we can form a crossed product order, given by $S * G = \sum_{\sigma\in G} Sx_\sigma$, with the usual rules of multiplication ($x_\sigma s = \sigma(s)x_\sigma$ for all $s \in S$, $\sigma \in G$ and $x_\sigma x_\tau = f(\sigma,\tau)x_{\sigma\tau}$). Since for all $s \in S$ and $\sigma \in G$, we have $(sx_\sigma)^{|G|} = \prod_{i=0}^{|G|-1} \sigma^i(s)f(\sigma^i,\sigma) \in S$, and S is integral over V, it follows from [**1**, Theorem 2.3] that $S * G$ is a V-order in $Q = K * G$. If $f = 1$, then $S * G$ becomes a *skew group ring*, denoted by $S \circ G$. Suppose that V has value group Γ, and that W is an extension of V in K with value group Δ. We say that (K,W) is *unramified* over (F,V) if $\Delta = \Gamma$ and $\overline{W}$ is separable over $\overline{V}$. If char$(\overline{V})$ does not divide $|\Delta/\Gamma|$ and $\overline{W}$ is separable over $\overline{V}$, we say that (K,W) is *tamely ramified*. Let g be the number of extensions of V in K, let $e = e(W \mid V) = |\Delta/\Gamma|$, and let $f = f(W \mid V) = [\overline{W} : \overline{V}]$. Although a 2-cocyle has also been denoted by the same symbol f, it should be clear from the context which is meant.

If $\overline{p}$ is the characteristic exponent of $\overline{V}$ (that is, $\overline{p} = 1$ if char$(\overline{V}) = 0$, else

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$\overline{p} = \text{char}(\overline{V})$), we have, in general, $[K : F] = efg\overline{p}^d$ for some non-negative integer d. We say that V is *defectless* in K if $[K : F] = efg$. If M is a maximal ideal of S, we may, without loss of generality, assume that $M = J(W) \cap S$. We call

$$G^Z(W \mid F) = \{\sigma \in G \mid \sigma(W) \subseteq W\} = \{\sigma \in G \mid \sigma(M) \subseteq M\}$$

the *decomposition group of W over F*, and

$$\begin{aligned} G^T(W \mid F) &= \{\sigma \in G \mid \sigma(x) - x \in J(W) \quad \forall x \in W\} \\ &= \{\sigma \in G \mid \sigma(x) - x \in M \quad \forall x \in S\} \end{aligned}$$

the *inertial group of W over F*. However, since K, F, G, W, S, V, M, and so on, will be fixed in this paper, we shall merely refer to the extension K/F as being 'unramified' or 'tamely ramified', and to $G^Z(W \mid F)$ as G^Z, and $G^T(W \mid F)$ as G^T, and so forth. Further, $n = |G|$, e, f, g, $\overline{p}$ and d are all fixed.

A lot of the theory of crossed product orders is known when V is a DVR. For example, in [**2**, Corollary A.5 and Proposition A.6] it was proved that $S \circ G$ is a maximal order if and only if (K, W) is unramified over (F, V), and in [**14**] it was shown that if K/F is tamely ramified, then $S * G$ is hereditary. The converse of the last statement, namely that if $S * G$ is hereditary then K/F is tamely ramified, does not hold (as will be shown in this paper) unless the residue ring $\overline{V}$ is perfect [**7**, Theorem 2] or the factor set f is trivial [**3**, Corollary 3.6]. In this paper, we aim to generalise these results to the case when V is not necessarily a DVR and the factor set f is not necessarily trivial. We shall employ recent results by Marubayashi and Yi [**10**, **15**] on the homological dimension of crossed products, and this will greatly simplify our proofs.

We recall that when V is a DVR, then any finite separable field extension is defectless (actually, we know that S is a finite free V-module by [**6**, Corollary 18.7]). When V is not necessarily a DVR, this need not be the case.

LEMMA 1. *Since K/F is a finite Galois extension, we have:*
(a) $(|G^T|, \overline{p}) = 1$ *if and only if K/F is tamely ramified and defectless, and*
(b) $|G^T| = 1$ *if and only if K/F is unramified and defectless.*

Proof. If $f = f_o\overline{p}^s$, where f_o is the degree over $\overline{V}$ of the maximal separable extension of $\overline{V}$ in $\overline{W}$, then $|G^T| = e\overline{p}^s\overline{p}^d$. So $(|G^T|, \overline{p}) = 1$ if and only if $\text{char}(\overline{V})$ does not divide e, and $\overline{p}^s = 1$ (that is, $\overline{W}$ is separable over $\overline{V}$), and $\overline{p}^d = 1$ (that is, K/F is defectless). The other assertion is proved in a similar manner, again using the fact that $|G^T| = e\overline{p}^s\overline{p}^d$.

We note that the 2-cocyle described above induces crossed product algebras $\overline{W} * G^Z$ and $\overline{W} * G^T$ in the obvious way.

LEMMA 2. *We have $S * G/(J(S) * G) \cong M_g(\overline{W} * G^Z)$. If $\overline{W} * G^Z$ is semisimple, then*
(a) *$J(S * G) = J(S) * G$, and*
(b) *$S * G$ is semihereditary.*

Proof. This follows from the proofs of [**10**, Lemma 2.3(ii) and Theorem 2.7].

THEOREM 1. *Let V be an arbitrary valuation ring. Then $S \circ G$ is semihereditary if and only if K/F is tamely ramified and defectless.*

Proof. Since S is a commutative semihereditary semilocal ring, the theorem is a restatement of [**10**, Theorem 2.9], in the light of Lemmas 1 and 2.

Given $x \in K$, we define its *trace* with respect to the extension K/F by the usual formula: $t_{K/F}(x) = \sum_{\sigma \in G} \sigma(x)$. Note that since V is integrally closed, $t_{K/F}(S) \subseteq V$. Now the proof given by Rosen [**13**, Theorem 40.13] carries over to the case when V is not necessarily a DVR, to establish that $S \circ G$ is semihereditary if and only if there exists in S an element of trace 1. Note the difference in terminology: what [**13**] refers to as *twisted group rings* are 'skew group rings' in our case, since the twisting is trivial; that is, $f = 1$. Our use of terminology is in line with that of [**12**].

Thus we obtain the following result of independent interest, which generalises [**3**, Theorem 3.2].

COROLLARY 1. *If (K, W) is a finite Galois extension of (F, V), and S is the integral closure of V in K, then K/F is tamely ramified and defectless if and only if $t_{K/F}(S) = V$.*

The following theorem is a generalisation of two classical results, given by Williamson and Harada. We shall prove the theorem by modifying, whenever necessary, the proofs of [**14**, Proposition 1.3] and [**7**, Theorem 2]. For the convenience of the reader, we present full details of the proof. We see, here and elsewhere, not only that the results in [**10**, **15**] enable us to extend the classical theory to the non-Noetherian setting, but also that, in some sense, we are able to obtain more simplified proofs of the old results as well.

THEOREM 2. *We have the following statements.*

(a) *If K/F is tamely ramified and defectless, then $J(S * G) = J(S) * G$ and $S * G$ is semihereditary.*

(b) *If $S * G$ is semihereditary and $\overline{V}$ is a perfect field, then K/F is tamely ramified and defectless, provided that V is a DVR or $J(V)$ is a non-principal ideal of V.*

Proof. (a) Note that since K/F is tamely ramified and defectless, $(|G^T|, \overline{p}) = 1$ by Lemma 1, and hence $\overline{W} * G^T$ is semisimple by [**14**, Theorem 1.1]. But $\overline{W} * G^T$ is a sub-crossed product of $\overline{W} * G^Z$, and both are Artinian rings. We therefore have $J(\overline{W} * G^Z) \cap (\overline{W} * G^T) = 0$.

Let $G^Z = \cup_i G^T \sigma_i$ be a right coset decomposition of G^Z with respect to G^T. Each element $b \in \overline{W} * G^Z$ can be expressed as

$$b = \sum_{i=1}^{t(b)} b_i, \quad \text{where} \quad b_i = \sum_{\gamma \in G^T} c_\gamma^{(i)} x_{\gamma\sigma_i} \quad \text{with} \quad c_\gamma^{(i)} \in \overline{W}.$$

Now let $b \in J(\overline{W} * G^Z)$, and write $b = \sum_{i=1}^{t(b)} b_i$. We claim that $b = 0$, and hence that $\overline{W} * G^Z$ is semisimple. The proof is by induction on $t(b)$. If $t(b) = 1$, then $b = b_1$. Therefore

$$b x_{\sigma_1^{-1}} \in J(\overline{W} * G^Z) \cap (\overline{W} * G^T) = 0,$$

and thus $b = 0$. Let $t(b) = t > 1$, and assume that $c = 0$ for each element $c \in J(\overline{W} * G^Z)$ for which $t(c) < t$. Since $\overline{W}$ is Galois over $\overline{V}$ in this case, we can write $\overline{W} = \overline{V}(\omega)$ for some $\omega \in \overline{W}$. Now let $a = \omega b - b\sigma_t^{-1}(\omega) \in J(\overline{W} * G^Z)$. Since

G^T acts trivially on $\overline{W}$, $a = \sum_{i=1}^{t-1}[\omega - \sigma_i(\sigma_t^{-1}(\omega))]b_i$. Since $t(a) < t$, we conclude from the induction hypothesis that $a = 0$. Since by the choice of the σ_i we have $\omega - \sigma_i(\sigma_t^{-1}(\omega)) \neq 0$ for $i \neq t$, it follows that $b_i = 0$ for $i \neq t$, and so $b = b_t$. Then $b = 0$, since $t(b_t) = 1$. So $\overline{W} * G^Z$ is semisimple, and hence $J(S * G) = J(S) * G$, and $S * G$ is semihereditary by Lemma 2.

(b) Assume that $\overline{p} > 1$, and fix $\sigma \in G^T$ of order $\overline{p}$ and set $P = \langle\sigma\rangle$. Since $P \leqslant G$, we know that $S * P$ is a semihereditary order in $K * P$ by [**10**, Proposition 2.2 (iv)]. Hence its overring in $K * P$, namely $W * P$, which exists since W is P-stable, is also semihereditary. But we have $W * P/(J(W) * P) \cong \overline{W} * P$, with P acting trivially on $\overline{W}$. By [**11**, Lemma 1.2.10], $\overline{W} * P$ is an ordinary group ring of P over $\overline{W}$; since $\overline{W}$ is a perfect field of characteristic $|P|$, being a finite extension of the perfect field $\overline{V}$. From [**4**, p. 435, Exercise 1], we conclude that $\overline{W} * P$ is completely primary. Since $J(W) * P \subseteq J(W * P)$ by [**12**, Theorem 4.2], we see that $W * P$ is completely primary as well. Hence, since $W * P$ is semihereditary, it must be an invariant valuation ring of $K * P$ and, in turn, $K * P$ is a division ring.

When V is a DVR, then the result follows from [**7**, Theorem 2]. Otherwise, let L be the fixed field of P, and let $U = W \cap L$. Then $Z(W * P) = U$. When $J(V)$ is a non-principal ideal of V, $J(U)$ is a non-principal ideal of U and hence, since $W * P$ is an invariant valuation ring, we find that $J(W * P) = J(U)(W * P) = J(W) * P$. But this means that the group ring $\overline{W} * P$ is semisimple, contradicting the fact that $|P| = \mathrm{char}(\overline{W})$.

So G^T contains no element of order $\overline{p}$. Therefore K/F is tamely ramified and defectless, by Lemma 1.

Remark. By combining Theorems 1 and 2(a), we have generalised the main theorem in [**14**, Section 1], which states that when V is a DVR, then $S * G$ is hereditary *for every factor set* f if and only if K/F is tamely ramified. We recall that in the classical setting, K/F is always defectless.

We shall see examples at the end of the paper which suggest that Theorem 2 may not be improved beyond its current form.

Corollary 2. *Let K/F be tamely ramified and defectless, and suppose that $J(V)$ is a non-principal ideal of V. Then*

(a) *$S * G$ is a semihereditary maximal order, and*

(b) *if V is rank-1, then $S \circ G \cong \mathrm{End}_V(S)$.*

Proof. We know that $S * G$ is semihereditary, and that $J(S * G) = J(S) * G$. Since $J(V)$ is not a principal ideal of V, $J(S) = J(V)S$, and thus $J(S * G) = J(V)(S * G)$, and so $S * G$ is a maximal order by [**8**, Lemma 3.6]. In particular, if V is rank-1 non-discrete, then $S \circ G$ is a maximal V-order. But $S \circ G \subseteq \mathrm{End}_V(S) \subseteq M_n(F)$. From [**5**, Section 3, Theorem 3], we conclude that $\mathrm{End}_V(S)$ is a V-order when V is rank-1: for fixed $x \in F \setminus V$. Let $\mathscr{F} = \{\text{subring } R \text{ of } M_n(F) \mid R \supseteq \mathrm{End}_V(S) \text{ and } x \notin R\}$. We observe that $\mathscr{F} \neq \emptyset$ since $\mathrm{End}_V(S) \in \mathscr{F}$, and that all elements of $\mathscr{F}$ are orders in $M_n(F)$ since they contain the order $S \circ G$. Zornify $\mathscr{F}$ via inclusion, and let T be a maximal element of $\mathscr{F}$. Then the order T is a maximal subring of $M_n(F)$, and hence a maximal V-order. It follows that $\mathrm{End}_V(S)$ is integral over V, and hence $S \circ G = \mathrm{End}_V(S)$ by the maximality of $S \circ G$.

REMARK. We suspect that, for any V, $\mathrm{End}_V(S)$ is a V-order in $M_n(F)$. If this is indeed the case, then $S \circ G \cong \mathrm{End}_V(S)$ whenever $S \circ G$ is semihereditary and $J(V)$ is not a principal ideal of V. When V is a principal ideal of V, one cannot always determine if $S * G$ is a maximal order, unless the cocycle f is explicitly known (see the examples below).

Recall that a V-order R of a central simple F-algebra, where V is a valuation ring of F, is *Azumaya over* V if it is a finitely generated V-module with $R/J(V)R$ a central simple $\overline{V}$-algebra.

THEOREM 3. *The order $S * G$ is Azumaya over V if and only if K/F is unramified and defectless.*

Proof. Suppose that $S * G$ is Azumaya over V. Then $J(S * G) = J(V)(S * G)$. But $J(V)(S*G) = (J(V)S)*G \subseteq J(S)*G \subseteq J(S*G)$. So we have $J(S*G) = J(S)*G$. Again, since $S * G$ is Azumaya, we have $n^2 = [\overline{S * G} : \overline{V}] = |M_g(\overline{W} * G^Z) : \overline{V}| = ef^2g^2\overline{p}^d$. Since $n = efg\overline{p}^d$, we immediately see that $e\overline{p}^d = 1$, and thus $|G^Z| = ef\overline{p}^d = f$.

Now, $G^T \leqslant G^Z$, and thus $\overline{W} * G^T$ is a sub-crossed product of $\overline{W} * G^Z$. Since G^T acts trivially on $\overline{W}$, we find that $\overline{W} * G^T \subseteq C_{\overline{W} * G^Z}(\overline{W})$, the centraliser of $\overline{W}$ in $\overline{W} * G^Z$. But $\overline{W} * G^Z$ is a central simple $\overline{V}$-algebra, since $M_g(\overline{W} * G^Z)$ is, as $S * G$ is Azumaya over V, and we now also know that $[\overline{W} : \overline{V}]^2 = [\overline{W} * G^Z : \overline{V}]$. So the field $\overline{W}$ is self-centralising in $\overline{W} * G^Z$. Therefore $\overline{W} * G^T = \overline{W}$, and thus $|G^T| = 1$. The result thus follows from Lemma 1.

Now suppose that K/F is unramified and defectless. So $\overline{W}$ is Galois over $\overline{V}$ and, since $|G^T| = 1$, $G^Z \cong \mathrm{Gal}(\overline{W}/\overline{V})$. But this isomorphism induces an isomorphism $\overline{W} * G^Z \cong \overline{W} * \mathrm{Gal}(\overline{W}/\overline{V})$, where $\overline{W} * \mathrm{Gal}(\overline{W}/\overline{V})$ is the classical crossed product algebra. So $S * G/(J(S) * G)$ is a simple ring, and $Z(S * G/(J(S) * G)) = \overline{V}$. By Lemma 2, $J(S * G) = J(S) * G$, and hence $S * G$ is primary and $Z(\overline{S * G}) = \overline{V}$. Now, since K/F is unramified, $J(S) = J(V)S$, and hence $J(S * G) = J(V)(S * G)$, and thus $S * G/(J(V)(S * G))$ is a central simple $\overline{V}$-algebra. But K/F is unramified and defectless. So S is a finite free V-module, by [**6**, Theorem 18.6]. Since $S * G$ is finite and free over S, we see that $S * G$ is finite and free over V.

We end by giving examples, in every characteristic, that exhibit some limitations to this theory of crossed product orders, but first we need the following proposition. The proposition also attempts to encapsulate the limitations of Theorem 2.

Recall that a ring is called *left coherent* if every finitely generated left ideal is finitely presented, and *right coherent* if every finitely generated right ideal is finitely presented. A ring is called *coherent* if it is both left and right coherent. A ring is called *left Bézout* if every finitely generated left ideal is principal. *Right Bézout* is defined similarly. A ring is called *Bézout* if it is both left and right Bézout.

We know that S is Bézout. Observe that since

$$[J(S)/J(V)S : V/J(V)] \leqslant [S/J(V)S : V/J(V)] < \infty,$$

there exist $s_1, s_2, \dots, s_m \in J(S)$ such that $J(S) = s_1S + s_2S + \cdots + s_mS + J(V)S$. So if $J(V)$ is a principal ideal of V, then $J(S)$ is a finitely generated ideal of S, and hence it is principal.

PROPOSITION. *Assume that* $\overline{p} > 1$.

(a) *Suppose that* K/F *has degree* $\overline{p}$, *and that* $J(V) = \pi V$, *a principal ideal of* V (*so* $J(S)$ *is a principal ideal of* (S)). *Assume* $J(S) = \pi S$. *Set* $K * G := (K/F, G, 1 - \pi)$. *Then either* $S * G$ *is Azumaya over* V *or, if this is not the case,* $K * G$ *is a division ring and* $S * G$ *is an invariant valuation ring of* $K * G$.

(b) *On the other hand, if* $S * G$ *is semihereditary and* $\overline{V}$ *is a perfect field, then either* K/F *is tamely ramified and defectless, or the following statements hold.*

(i) $J(V)$ *is a principal ideal of* V.

(ii) *For any* $P \leqslant G^T$ *with* $|P| = \overline{p}$, $K * P$ *is a division algebra and* $S * P$ *is an invariant valuation ring of* $K * P$.

(iii) *If* L *is the fixed field of* P, *then there exists some* $\pi \in S \cap L$ *such that* $J(S) = \pi S$ *and* $K * P \cong (K/L, P, 1 - \pi)$.

Proof. (a) Let $G = \langle \sigma \rangle$. If $|G^T| = 1$, then $S * G$ is Azumaya over V, by Theorem 3; otherwise, $|G^T| = \overline{p}$ and $S * G$ is not Azumaya over V. In the latter case, we have $G = G^T$ and $S = W$. Since W is a coherent ring, and $W * G$ is a free left and right W-module of finite rank, the proof of [**15**, Theorem 3.2] shows that $W * G$ is coherent. Note that $\overline{W} * G$ is an ordinary group ring and hence, by [**4**, p. 435, Exercise 1], $J(\overline{W} * G) = \sum_{i=1}^{\overline{p}-1}(1 - \overline{x}_{\sigma^i})(\overline{W} * G)$, and $\overline{W} * G$ is completely primary. But $\pi = (1 - x_\sigma)(1 + x_\sigma + x_{\sigma^2} + \cdots + x_{\sigma^{-1}}) \in (1 - x_\sigma)(W * G)$. Also, for $2 \leqslant i \leqslant \overline{p} - 1$, we have $1 - x_{\sigma^i} = 1 - x_\sigma^i = (1 - x_\sigma)(1 + x_\sigma + x_{\sigma^2} + \cdots + x_{\sigma^{i-1}}) \in (1 - x_\sigma)(W * G)$. Therefore $J(W) * G = \pi(W * G) \subseteq (1 - x_\sigma)(W * G)$, and $J(W * G) = (1 - x_\sigma)(W * G)$, a free $W * G$-module. Hence the V-order $W * G$ is semihereditary, by [**9**, Theorem 2.7]. Further, since $W * G$ is also completely primary, we conclude that it must be an invariant valuation ring of $K * G$, and in turn $K * G$ is a division algebra.

(b) Parts (i) and (ii) follow from Theorem 2(b) and its proof. For part (iii), note that $K * P \cong (K/L, P, a)$ for some $a \in U(S) \cap L$. By the proof of [**7**, Theorem 2], which holds even when the valuation ring is not a DVR, $J(S) = (1 - a)S$. We set $\pi = 1 - a$.

EXAMPLE 1. Let $F = \mathbb{Q}(t)$, a function field in one variable over the field of the rationals. Let U_1 be the t-adic valuation ring of F, and set $V = \mathbb{Z}_2 + J(U_1)$, where $\mathbb{Z}_2$ is the 2-adic valuation ring of $\mathbb{Q}$. Let $K = F(\sqrt{t})$, a cyclic extension of F with group $G = \langle \sigma \rangle$. Let W be the unique extension of V to K. Clearly, $\overline{W} = \overline{V} = \mathbb{Z}/2\mathbb{Z}$, and $e(W \mid V) = 2$. Let $f \in Z^2(G, U(W))$ be defined by $f(\sigma, \sigma) = -1$, $f(1, \sigma) = f(\sigma, 1) = f(1, 1) = 1$.

From the proposition (with $\pi = 2$), it follows that $S * G$ is an invariant valuation ring of $K * G$, and hence is semihereditary. However, although $\overline{V}$ is a perfect field, K/F is not tamely ramified.

Now let W_1 be the unique extension of U_1 to K. Then $e(W_1 \mid U_1) = 2$. We see that while $W_1 * G$ is a maximal U_1-order, in the classical sense of the term, (K, W_1) is not unramified over (F, U_1), and $W_1 \circ G$ is not a maximal U_1-order.

The following example, communicated to the author by P. Morandi, illustrates a similar phenomenon in positive characteristic.

EXAMPLE 2. Let L be a field of characteristic $p > 0$, let $F = L((x))((y))$ be the iterated Laurent series field in two variables over L, and let $K = F(t)$ be the cyclic extension of F satisfying $t^p - t = 1/y$, with group $G = \langle \sigma \rangle$. Let V be the standard rank-2 valuation ring of F, and W the extension of V to K. It is easily seen that $e(W \mid V) = p$. Let $f \in Z^2(G, U(W))$ be defined by $f(\sigma^i, \sigma^j) = 1$ for $i + j < p$, and by $f(\sigma^i, \sigma^j) = 1 - x$ otherwise, where $0 \leqslant i, j < p$.

It then follows from the proposition (with $\pi = x$) that $S * G$ is an invariant valuation ring of $K * G$, and hence is semihereditary. But K/F is not tamely ramified. Note that we may choose $L\ (= \overline{V})$ to be a perfect field.

Let U_1 be the DVR of F containing V, and let W_1 be the extension of U_1 to K. Then $W_1 * G$ is an invariant valuation ring, being an overring of $S * G$ in $K * G$. Observe that $e(W_1 \mid U_1) = p$, and so (K, W_1) is not tamely ramified over (F, U_1). Thus the converse of the result given by Williamson [**14**, Proposition 1.4] does not hold, and thus we may not drop the perfectness assumption in Harada's result [**7**, Theorem 2]. Also, note that while $W_1 * G$ is a classical maximal order, $W_1 \circ G$ is not even hereditary.

We conclude that properties of the order $S * G$ cannot always be solely determined by the nature of the extension K/F, but that one has to consider the 2-cocycle f as well. The converse is also true.

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