

On a class of semihereditary crossed-product orders

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Abstract

Let F be a field, let V be a valuation ring of F of arbitrary Krull dimension (rank), let K be a finite Galois extension of F with group G , and let S be the integral closure of V in K . Let $f : G \times G \mapsto K \setminus \{0\}$ be a normalized two-cocycle such that $f(G \times G) \subseteq S \setminus \{0\}$, but we do not require that f should take values in the group of multiplicative units of S . One can construct a crossed-product V -algebra $A_f = \sum_{\sigma \in G} Sx_\sigma$ in a natural way, which is a V -order in the crossed-product F -algebra $(K/F, G, f)$. If V is unramified and defectless in K , we show that A_f is semihereditary if and only if for all $\sigma, \tau \in G$ and every maximal ideal M of S , $f(\sigma, \tau) \notin M^2$. If in addition $J(V)$ is not a principal ideal of V , then A_f is semihereditary if and only if it is an Azumaya algebra over V .

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1 Introduction

In this paper we study certain orders over valuation rings in central simple algebras. If R is a ring, then $J(R)$ will denote its Jacobson radical, $U(R)$ its group of multiplicative units, and $R^\#$ the subset of all the non-zero elements. The residue ring $R/J(R)$ will be denoted by \overline{R} . Given the ring R , it is called *primary* if $J(R)$ is a maximal ideal of R . It is called *hereditary* if one-sided ideals are projective R -modules. It is called *semihereditary* (respectively *Bézout*) if finitely generated one-sided ideals are projective R -modules (respectively are principal). Let V be a valuation ring of a field F . If Q is a finite-dimensional central simple F -algebra, then a subring R of Q is called

an order in Q if $RF = Q$. If in addition $V \subseteq R$ and R is integral over V , then R is called a V -order. If a V -order R is maximal among the V -orders of Q with respect to inclusion, then R is called a maximal V -order (or just a maximal order if the context is clear). A V -order R of Q is called an *extremal* V -order (or simply *extremal* when the context is clear) if for every V -order B in Q with $B \supseteq R$ and $J(B) \supseteq J(R)$, we have $B = R$. If R is an order in Q , then it is called a *Dubrovin valuation ring* of Q (or a *valuation ring* of Q in short) if it is semihereditary and primary (see [1, 2]).

In this paper, V will denote a commutative valuation ring of *arbitrary* Krull dimension (rank). Let F be its field of quotients, let K/F be a finite Galois extension with group G , and let S be the integral closure of V in K . If $f \in Z^2(G, U(K))$ is a normalized two-cocycle such that $f(G \times G) \subseteq S^\#$, then one can construct a “crossed-product” V -algebra

$$A_f = \sum_{\sigma \in G} Sx_\sigma,$$

with the usual rules of multiplication ($x_\sigma s = \sigma(s)x_\sigma$ for all $s \in S, \sigma \in G$ and $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$). Then A_f is associative, with identity $1 = x_1$, and center $V = Vx_1$. Further, A_f is a V -order in the crossed-product F -algebra $\Sigma_f = \sum_{\sigma \in G} Kx_\sigma = (K/F, G, f)$. Following [4], we let $H = \{\sigma \in G \mid f(\sigma, \sigma^{-1}) \in U(S)\}$. Then H is a subgroup of G .

In this paper, we will *always* assume that V is unramified and defectless in K (for the definitions of these terms, see [3]). By [3, Theorem 18.6], S is a finitely generated V -module, hence A_f is always finitely generated over V . If V_1 is a valuation ring of K lying over V then $\{\sigma \in G \mid \sigma(x) - x \in J(V_1) \forall x \in V_1\}$ is called the *inertial group* of V_1 over F . By [10, Lemma 1], the condition that V is unramified and defectless in K is equivalent to saying that the inertial group of V_1 over F is trivial, since K/F is a finite Galois extension.

These orders were first studied in [4], and later in [6] and [11]. In [4] and [11], only the case when V is a discrete valuation ring (DVR) was considered. In [11], hereditary properties of crossed-product orders were examined. In [4] and [6], valuation ring properties of the crossed-product orders were explored, and the latter considered the cases when either V had arbitrary Krull dimension but was indecomposed in K , or V was a discrete finite rank valuation ring, that is, its value group is $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. When V is a DVR, then any V -order in Σ_f containing S is a crossed-product order of the form A_g for some two-cocycle $g : G \times G \mapsto S^\#$, with g cohomologous to f over K ,

by [4, Proposition 1.3], but this need not be the case in general. While [6] considered any V -order in Σ_f containing S , some of which were not of the type described above and so in that sense its scope was wider than ours, in this paper we shall only be concerned with crossed-product orders A_g where g is either f (almost always), or is cohomologous to f over K , that is, if there are elements $\{c_\sigma \mid \sigma \in G\} \subseteq K^\#$ such that $g(\sigma, \tau) = c_\sigma \sigma(c_\tau) c_{\sigma\tau}^{-1} f(\sigma, \tau)$ for all $\sigma, \tau \in G$, a fact denoted by $g \sim_K f$.

The purpose of this paper is to generalize the results of [11] to the case when V is not necessarily a DVR. The main results of this paper are as follows: A_f is semihereditary if and only if for all $\sigma, \tau \in G$ and every maximal ideal M of S , $f(\sigma, \tau) \notin M^2$; if $J(V)$ is not a principal ideal of V , then A_f is semihereditary if and only if it is an Azumaya algebra over V . As in [11], the utility of these criteria lie in their simplicity.

Although in our case the valuation ring V need not be a DVR, some of the steps in the proofs in [4] and [11] remain valid, *mutatis mutandis*, owing to the theory developed in [8, 9]. We shall take full advantage of this whenever the opportunity arises. Aside from the difficulties inherit when dealing with V -orders that are not necessarily noetherian, the hurdles encountered in this theory arise mainly due to the fact that the two-cocycle f is not assumed to take on values in $U(S)$.

2 Preliminaries

In this section, we gather together various results that will help us prove the main results of this paper, which are in the next section. For the convenience of the reader, we have included complete proofs whenever it warrants, although the arguments are sometimes routine.

The following lemma is essentially embedded in the proof of [8, Proposition 1.8], and the remark that follows it.

Lemma 2.1. *Let A be a finitely generated extremal V -order in a finite-dimensional central simple F -algebra Q .*

1. *If B is a V -order of Q containing A , then B is also a finitely generated extremal order. If in addition B is a maximal V -order, then it is a valuation ring of Q .*
2. *If W is an overring of V in F with $V \subsetneq W$, then WA is a valuation ring of Q with center W .*

Proof. Let B be a V -order containing A . By [8, Proposition 1.8], A is semihereditary, hence B is semihereditary by [12, Lemma 4.10], and therefore B is extremal by [8, Theorem 1.5]. Since $[B/J(B) : V/J(V)] \leq [\Sigma_f : F] < \infty$, there exists $a_1, a_2, \dots, a_m \in B$ such that $B = a_1V + a_2V + \dots + a_mV + J(B)$. But by [8, Proposition 1.4], $J(B) \subseteq J(A)$, since A is extremal. Therefore $B = a_1V + a_2V + \dots + a_mV + A$, a finitely generated V -order. If, in addition B is a maximal V -order, then by the remark after [8, Proposition 1.8], B is a valuation ring of Q .

Now let W be a proper overring of V in F . Let C be a maximal V -order containing A . Then C is a valuation ring of Q , as seen above, hence WC is a valuation ring of Q with center W . Since A is an extremal V -order, we have $J(C) \subseteq J(A)$, thus $WC = WJ(V)C \subseteq WJ(C) \subseteq WA \subseteq WC$, so that $WA = WC$. Thus WA is always a valuation ring of Q . \square

Since A_f is finitely generated over V , we immediately have the following lemma, because of [8, Proposition 1.8], the remark that follows it, and the fact that Bézout V -orders are maximal orders by [12, Theorem 3.4].

Lemma 2.2. *Given the crossed-product order A_f ,*

1. *it is an extremal order if and only if it is semihereditary.*
2. *it is a maximal order if and only if it is a valuation ring, if and only if it is Bézout.*

Lemma 2.3. *Let W be a valuation ring of F such that $V \subsetneq W$, and let $R = WS$.*

1. *Then R is the integral closure of W in K , and W is also unramified and defectless in K .*
2. *Let $t \in S$ satisfy $t \notin M^2$ for every maximal ideal M of S . Then $t \in U(R)$. If in addition $J(V)$ is a non-principal ideal of V , then $t \in U(S)$.*

Proof. The ring R is obviously integral over W . Since it contains S , it is also integrally closed in K , hence it is the integral closure of W in K .

Now let $V_1 \subseteq W_1$ be valuation rings of K lying over V and W respectively. Then $J(W_1) \subseteq J(V_1)$, hence the inertial group of W_1 over F , $\{\sigma \in G \mid \sigma(x) - x \in J(W_1) \forall x \in W_1\}$, is contained in the inertial group of V_1 over F , $\{\sigma \in G \mid \sigma(x) - x \in J(V_1) \forall x \in V_1\}$. Since V is unramified and defectless

in K , the latter group is trivial, forcing W to be unramified and defectless in K .

Let W_1 be a valuation ring of K lying over W , and let V_1 be a valuation ring of K lying over V such that $V_1 \subseteq W_1$, as in the preceding paragraph. Let $M = J(V_1) \cap S$, a generic maximal ideal of S . We claim that $M^2 = J(V_1)^2 \cap S$. To see this, note that $M^2 = (J(V_1) \cap S)(J(V_1) \cap S) \subseteq J(V_1)^2 \cap S$, and $M^2 V_1 = (J(V_1) \cap S)(J(V_1) \cap S)V_1 = J(V_1)^2 = (J(V_1)^2 \cap S)V_1$. If V' is an extension of V to K different from V_1 , then $M^2 V' = V' = (J(V_1)^2 \cap S)V'$. Thus $M^2 = J(V_1)^2 \cap S$ as desired. If $t \in S$ satisfies $t \notin M^2$, then $t \notin J(V_1)^2$. Since $J(W_1) \not\subseteq J(V_1)^2$, we have $t \in U(W_1)$. Since W_1 was an arbitrary extension of W in K , we conclude that $t \in U(R)$. If $J(V)$ is a non-principal ideal of V , then $J(V_1)^2 = J(V_1)$, hence $t \in U(V_1)$ for every such extension V_1 of V to K , and we conclude that $t \in U(S)$. \square

Part 4 of the following lemma was originally proved in [4] when V is a DVR. The same arguments work when V is an arbitrary valuation ring.

Lemma 2.4. *Given a $\sigma \in G$, let $I_\sigma = \cap M$, where the intersection is taken over those maximal ideals M of S for which $f(\sigma, \sigma^{-1}) \notin M$. Then*

1. $I_\sigma = \{x \in S \mid xf(\sigma, \sigma^{-1}) \in J(V)S\}$.
2. $I_\sigma^{\sigma^{-1}} = I_{\sigma^{-1}}$.
3. *If $f(\sigma, \sigma^{-1}) \notin M^2$ for every maximal ideal M of S , then $I_\sigma f(\sigma, \sigma^{-1}) = J(V)S$.*
4. $J(A_f) = \sum_{\sigma \in G} I_\sigma x_\sigma$.

Proof. Let $x \in S$. Clearly, if $x \in I_\sigma$ then $xf(\sigma, \sigma^{-1}) \in J(V)S$. On the other hand, if $x \notin I_\sigma$ then there exists a maximal ideal M of S such that $x, f(\sigma, \sigma^{-1}) \notin M$, hence $xf(\sigma, \sigma^{-1}) \notin M$, and thus $xf(\sigma, \sigma^{-1}) \notin J(V)S$.

The second statement is proved in the same manner as [11, Sublemma]. To see that the third statement holds, we note that $I_\sigma f(\sigma, \sigma^{-1}) \subseteq J(V)S$. We claim that $I_\sigma f(\sigma, \sigma^{-1}) = J(V)S$. To see this, let M be a maximal ideal of S . If $f(\sigma, \sigma^{-1}) \notin M$, then $(I_\sigma f(\sigma, \sigma^{-1}))S_M = J(S_M) = (J(V)S)S_M$. On the other hand, if $f(\sigma, \sigma^{-1}) \in M$ then, since $f(\sigma, \sigma^{-1}) \notin M^2$, we have $J(S_M)^2 \not\subseteq I_\sigma f(\sigma, \sigma^{-1})S_M \subseteq J(S_M)$, hence $I_\sigma f(\sigma, \sigma^{-1})S_M = J(S_M) = (J(V)S)S_M$, and thus $I_\sigma f(\sigma, \sigma^{-1}) = J(V)S$. By [6, Lemma 1.3], $J(A_f) = \sum_{\sigma \in G} (J(A_f) \cap Sx_\sigma)$. Therefore the fourth statement can be verified in exactly the same manner as [4, Proposition 3.1(b)], because of the observations made above. \square

The following lemma is a generalization of [4, Proposition 1.3].

Lemma 2.5. *Let $B \subseteq \Sigma_f$ be a V -order. There is a normalized cocycle $g : G \times G \mapsto S^\#$, $g \sim_K f$, such that $B = A_g$ (viewed as a subalgebra of Σ_f in a natural way) if and only if $B \supseteq S$ and B is finitely generated over V . When this occurs, $B = \sum_{\sigma \in G} S k_\sigma x_\sigma$ for some $k_\sigma \in K^\#$.*

Proof. Suppose $B \supseteq S$. By [6, Lemma 1.3], $B = \sum_{\sigma \in G} B_\sigma x_\sigma$, where each B_σ is a non-zero S -submodule of K . If in addition B is finitely generated over V , then each B_σ is finitely generated over V : if $B = \sum_{i=1}^n V y_i$ then, if we write $y_i = \sum_{\tau \in G} k_\tau^{(i)} x_\tau$ with $k_\tau^{(i)} \in K$, we see that B_σ is generated by $\{k_\tau^{(i)}\}_{i=1}^n$ over V . Since S is a commutative Bézout domain with K as its field of quotients, $B_\sigma = S k_\sigma$ for some $k_\sigma \in K^\#$. Thus we get $B = \sum_{\sigma \in G} S k_\sigma x_\sigma$. Since B is integral over V , $B_1 = S$ and so we can choose $k_1 = 1$. Define $g : G \times G \mapsto S^\#$ by $g(\sigma, \tau) k_{\sigma\tau} x_{\sigma\tau} = (k_\sigma x_\sigma)(k_\tau x_\tau)$, as in [4, Proposition 1.3]. Since $k_1 = 1$, g is also a normalized two-cocycle. The converse is obvious. \square

Lemma 2.6. *Suppose S is a valuation ring of K . Then the following are equivalent:*

1. $J(V)A_f$ is a maximal ideal of A_f .
2. $H = G$.
3. A_f is Azumaya over V .

Proof. Suppose $J(V)A_f$ is a maximal ideal of A_f . Note that $A_f/J(V)A_f = \sum_{\sigma \in G} \overline{S} \tilde{x}_\sigma$. By [5, Theorem 10.1(c)], $J = \sum_{\sigma \notin H} \overline{S} \tilde{x}_\sigma$ is an ideal of $A_f/J(V)A_f$. Since $A_f/J(V)A_f$ is simple, $J = 0$, hence $H = G$. \square

We set up additional notation, following [4] and [11]. Let L be an intermediate field of F and K , let G_L be the Galois group of K over L , let U be a valuation ring of L lying over V , and let T be the integral closure of U in K . Then one can obtain a two-cocycle $f_{L,U} : G_L \times G_L \mapsto T^\#$ from f by restricting f to $G_L \times G_L$, and embedding $S^\#$ in $T^\#$. As before, $A_{f_{L,U}} = \sum_{\sigma \in G_L} T x_\sigma$ is a U -order in $\Sigma_{f_{L,U}} = \sum_{\sigma \in G_L} K x_\sigma = (K/L, G_L, f_{L,U})$, and U is unramified and defectless in K . If M is a maximal ideal of S , and L is the decomposition field of M and $U = L \cap S_M$, then we will denote $f_{L,U}$ by f_M , $A_{f_{L,U}}$ by A_{f_M} , $\Sigma_{f_{L,U}}$ by Σ_{f_M} , L by K_M , and the decomposition group G_L by D_M , as in [4]. Further, we let $H_M = \{\sigma \in D_M \mid f_M(\sigma, \sigma^{-1}) \in U(S_M)\}$, a subgroup of D_M .

Given a maximal ideal M of S , let $M = M_1, M_2, \dots, M_r$ be the complete list of maximal ideals of S , let $U_i = S_{M_i} \cap K_{M_i}$ with $U = U_1$, and let (K_i, S_i) be a Henselization of (K, S_{M_i}) . Let (F_h, V_h) be the unique Henselization of (F, V) contained in (K_1, S_1) . We note that (F_h, V_h) is also a Henselization of (K_M, U) . By [7, Proposition 11], we have $S \otimes_V V_h \cong S_1 \oplus S_2 \oplus \dots \oplus S_r$.

Part (1) of the following lemma was originally proved in [4] in the case when V is a DVR. Virtually the same proof holds in the general case. Part (2)(c) is a generalization of [4, Corollary 3.11].

Lemma 2.7. *With the notation as above, we have*

1. *the crossed-product order A_f is primary if and only if for every maximal ideal M of S there is a set of right coset representatives g_1, g_2, \dots, g_r of D_M in G (i.e., G is the disjoint union $\cup_i D_M g_i$) such that for all i , $f(g_i, g_i^{-1}) \notin M$.*
2. *if the crossed-product order A_f is primary, then*
 - (a) *$A_f \otimes_V V_h \cong M_r(A_{f_M} \otimes_U V_h)$, hence*
 - (b) *$A_f/J(A_f) \cong M_r(A_{f_M}/J(A_{f_M}))$, and*
 - (c) *A_f is a valuation ring of Σ_f if and only if A_{f_M} is a valuation ring of Σ_{f_M} for some maximal ideal M of S . When this occurs, A_{f_M} is a valuation ring of Σ_{f_M} for every maximal ideal M of S .*
 - (d) *A_f is Azumaya over V if and only if $H_M = D_M$ for some maximal ideal M of S . When this occurs, $H_M = D_M$ for every maximal ideal M of S .*

Proof. The proof of [4, Theorem 3.2], appropriately adapted, works here as well to establish part (1). We outline the argument, for the convenience of the reader: For a $\sigma \in G$, let I_σ be as in Lemma 2.4, and, for a maximal ideal M of S , set $\hat{M} := \cap_{N \text{ max, } N \neq M} N$. If I is an ideal of A_f then, by [6, Lemma 1.3], $I = \sum_{\sigma \in G} (I \cap Sx_\sigma)$, so A_f is primary if and only if the following condition holds: if $\sigma \in G$ and T is an ideal of S such that $T \not\subseteq I_\sigma$, then $A_f T x_\sigma A_f = A_f$.

If A_f is primary and M is a maximal ideal of S , then $A_f = A_f \hat{M} x_1 A_f$. Therefore if $G = \cup_{j=1}^r h_j D_M$ is a left coset decomposition, then

$$S = \sum_j \hat{M}^{h_j} \left(\sum_{d \in D_M} f(h_j d, d^{-1} h_j^{-1}) \right)$$

as in the proof of [4, Theorem 3.2], so that, if we fix i , $1 \leq i \leq r$, and localize at M^{h_i} , we get

$$S_{M^{h_i}} = \sum_{j \neq i} J(S_{M^{h_j}}) \left(\sum_{d \in D_M} f(h_j d, d^{-1} h_j^{-1}) \right) + S_{M^{h_i}} \left(\sum_{d \in D_M} f(h_i d, d^{-1} h_i^{-1}) \right),$$

and hence $\sum_{d \in D_M} f(h_i d, d^{-1} h_i^{-1}) \notin M^{h_i}$. So there is an element $d_i \in D_M$ such that $f(h_i d_i, d_i^{-1} h_i^{-1}) \notin M^{h_i}$. Let $g_i = d_i^{-1} h_i^{-1}$. Then g_1, g_2, \dots, g_r have the desired properties.

For the converse, suppose $\sigma \in G$ and T is an ideal of S such that $T \not\subseteq I_\sigma$. We need to show that $A_f T x_\sigma A_f = A_f$. Since $T \not\subseteq I_\sigma$, there is a maximal ideal M of S such that $f(\sigma, \sigma^{-1}) \notin M$ and $T \not\subseteq M$. The argument in [4, Theorem 3.2] shows that $A_f T x_\sigma A_f \supseteq \sum_{i=1}^r T_i$, where $T_i = T^{g_i^{-1}} f^{g_i^{-1}}(\sigma, \sigma^{-1} g_i) f(g_i^{-1}, g_i)$ are ideals of S satisfying the condition $T_i \not\subseteq M^{g_i^{-1}}$. Inasmuch as $g_1^{-1}, g_2^{-1}, \dots, g_r^{-1}$ form a complete set of *left* coset representatives of D_M in G , the ideal $\sum_{i=1}^r T_i$ is not contained in any maximal ideal of S . Therefore $\sum_{i=1}^r T_i = S$, and so $A_f T x_\sigma A_f = A_f$.

Using part (1) and the fact that $S \otimes_V V_h \cong S_1 \oplus S_2 \oplus \dots \oplus S_r$, we can construct a full set of matrix units in $A_f \otimes_V V_h$ and hence verify part (2)(a), as in the proof of [4, Theorem 3.12] (see also the remark after [4, Theorem 3.12]). Part (2)(b) follows from (2)(a) and [8, Lemma 3.1]; part (2)(c) follows from (2)(a); and (2)(d) follows from (2)(a) and Lemma 2.6. \square

3 The Main Results

We now give the main results of this paper. There are essentially two parallel theories: one takes effect when $J(V)$ is a principal ideal of V , and the other when it is not. In the former case, the order A_f displays characteristics akin to the situation when V is a DVR. Our theory, however, yields surprising results in the latter case. It turns out in this case that the property that A_f is Azumaya over V is equivalent to a much more weaker property: that it is an extremal V -order in Σ_f .

Proposition 3.1. *The order A_f is Azumaya over V if and only if $H = G$.*

Proof. Suppose A_f is Azumaya over V . Let M be a maximal ideal of S . By Lemma 2.7(1), there is a set of right coset representatives g_1, g_2, \dots, g_r of

D_M in G such that $f(g_i, g_i^{-1}) \notin M$. If $\sigma \in G$, then $\sigma = hg_i$ for some $h \in D_M$ and some i . Since A_f is Azumaya, $H_M = D_M$ by Lemma 2.7(2)(d), hence we have $f(h^{-1}, h) \notin M$. Because

$$f^{h^{-1}}(hg_i, g_i^{-1}h^{-1})f^{h^{-1}}(h, g_i)f^{g_i}(g_i^{-1}, h^{-1}) = f(h^{-1}, h)f(g_i, g_i^{-1}),$$

we conclude that $f(\sigma, \sigma^{-1}) \notin M$. Since M is arbitrary, $f(\sigma, \sigma^{-1}) \in U(S)$ for every $\sigma \in G$, so that $H = G$.

The converse is well-known and straightforward to demonstrate. □

It is perhaps instructive to compare the above proposition to [10, Theorem 3].

Recall that $J(V)$ is a non-principal ideal of V if and only if $J(V)^2 = J(V)$.

Proposition 3.2. *Suppose $J(V)$ is a non-principal ideal of V . Then the following statements about the crossed-product order A_f are equivalent:*

1. A_f is an extremal V -order in Σ_f .
2. A_f is a semihereditary V -order.
3. A_f is a maximal V -order in Σ_f .
4. A_f is a Bézout V -order.
5. A_f is a valuation ring of Σ_f .
6. A_f is Azumaya over V .

Proof. By Lemma 2.2, it suffices to demonstrate that (1) \implies (5) \implies (6). So suppose A_f is an extremal V -order. Let B be a maximal V -order containing A_f . By Lemma 2.1, B is a valuation ring finitely generated over V . By Lemma 2.5, we get that $B = \sum_{\sigma \in G} S k_\sigma x_\sigma$ for some $k_\sigma \in K^\#$. Since A_f is extremal, we have $J(B) \subseteq J(A_f)$ by [8, Proposition 1.4], so $J(V)B \subseteq A_f$. Therefore $\sum_{\sigma \in G} J(S)k_\sigma x_\sigma = J(V)B = J(V)^2 B \subseteq J(V)A_f = \sum_{\sigma \in G} J(S)x_\sigma$, so that $J(S)k_\sigma \subseteq J(S)$. Hence for each maximal ideal M of S , we have $S_M J(S)k_\sigma \subseteq S_M J(S)$, that is, $J(S_M)k_\sigma \subseteq J(S_M)$. This shows that $k_\sigma \in S_M$ for all M and so $k_\sigma \in S$ for every $\sigma \in G$, and thus $A_f = B$, a valuation ring.

Now suppose A_f is a valuation ring of Σ_f . By Lemma 2.7(2), to show that A_f is Azumaya over V , we may as well assume S is a valuation ring of

K . By [2, §2, Theorem 1], $J(A_f) = J(V)A_f$, and so A_f is Azumaya over V by Lemma 2.6. □

Remark. It follows from Lemma 2.3(2) and Proposition 3.1 that, if $J(V)$ is a non-principal ideal of V , then the crossed-product order A_f is extremal if and only if for all $\tau, \gamma \in G$ and every maximal ideal M of S , $f(\tau, \gamma) \notin M^2$.

If W is a valuation ring of F such that $V \subsetneq W$, then we will denote by B_f the W -order $WA_f = \sum_{\sigma \in G} Rx_\sigma$, where $R = \overline{WS}$ is the integral closure of W in K by Lemma 2.3. Recall that W is also unramified and defectless in K .

Proposition 3.3. *Suppose $J(V)$ is a principal ideal of V . Then A_f is semihereditary if and only if for all $\tau, \gamma \in G$ and every maximal ideal M of S , $f(\tau, \gamma) \notin M^2$.*

Proof. The result holds when the Krull dimension of V is one, by [11, Corollary], since V is a DVR in this case. So let us assume from now on that the Krull dimension of V is greater than one.

Let $p = \bigcap_{n \geq 1} J(V)^n$. Then p is a prime ideal of V , $W = V_p$ is a minimal overring of V in F , and $\tilde{V} = V/J(W)$ is a DVR of \overline{W} . Set $B_f = WA_f$, as above.

Suppose A_f is semihereditary. We will show that for each $\tau \in G$ and each maximal ideal M of S , $f(\tau, \tau^{-1}) \notin M^2$.

First, assume that V is indecomposed in K . By [6, Proposition 2.6], A_f is primary, hence it is a valuation ring of Σ_f . Therefore B_f is Azumaya over W , by [6, Proposition 2.10], and $f(G \times G) \subseteq U(R)$, by Proposition 3.1. Observe that R is a valuation ring of K lying over W and \overline{R} is Galois over \overline{W} , with group G , and $B_f/J(B_f) = \sum_{\sigma \in G} \overline{R}\tilde{x}_\sigma$ is a crossed-product \overline{W} -algebra. Further, $A_f/J(B_f)$ has center \tilde{V} , a DVR of \overline{W} , and is a crossed-product \tilde{V} -order in $B_f/J(B_f)$ of the type under consideration in this paper, since \tilde{V} is unramified in \overline{R} and $f(G \times G) \subseteq S \cap U(R)$. Since the crossed-product \tilde{V} -order $A_f/J(B_f)$ is a valuation ring of $B_f/J(B_f)$ hence hereditary, it follows from [11, Theorem] that for each $\tau \in G$, $f(\tau, \tau^{-1}) \notin J(S)^2$.

Suppose V is not necessarily indecomposed in K , but assume A_f is a valuation ring. Fix a maximal ideal M of S . By Lemma 2.7(1), there is a set of right coset representatives g_1, g_2, \dots, g_r of D_M in G such that $f(g_i, g_i^{-1}) \notin M$. If $\tau \in G$, then $\tau = hg_i$ for some $h \in D_M$ and some i .

By Lemma 2.7(2), A_{f_M} is a valuation ring of Σ_{f_M} . Hence, by the preceding paragraph, $f_M(h^{-1}, h) \notin M^2$, and thus $f(h^{-1}, h) \notin M^2$. But the following holds:

$$f^{h^{-1}}(hg_i, g_i^{-1}h^{-1})f^{h^{-1}}(h, g_i)f^{g_i}(g_i^{-1}, h^{-1}) = f(h^{-1}, h)f(g_i, g_i^{-1}).$$

Therefore we must have $f(\tau, \tau^{-1}) \notin M^2$.

Now suppose that A_f is not necessarily a valuation ring. To show that for each $\tau \in G$ and each maximal ideal M of S we have $f(\tau, \tau^{-1}) \notin M^2$, one only needs to emulate the corresponding steps in the proof of [11, Theorem], equipped with the following four observations: 1) Any maximal V -order containing A_f is a valuation ring, by Lemma 2.1, hence A_f is the intersection of finitely many valuation rings all with center V , since $J(V)$ is a principal ideal of V , by [9, Theorem 2.5]. 2) If B is one such valuation ring containing A_f , then $B = A_g = \sum_{\tau \in G} Sk_\tau x_\tau$ for some $k_\tau \in K^\#$, where $g : G \times G \mapsto S^\#$ is some normalized two-cocycle, by Lemma 2.1(1) and Lemma 2.5. Fix $\sigma \in G$ and a maximal ideal N of S . We may choose B such that $k_\sigma \in U(S_N)$, as in the proof of [11, Theorem]. 3) Both $J(A_f)$ and $J(A_g)$ are as in Lemma 2.4, that is, $J(A_f) = \sum_{\sigma \in G} I_\sigma x_\sigma$ (respectively $J(B_f) = \sum_{\sigma \in G} J_\sigma k_\sigma x_\sigma$) where $I_\sigma = \cap M$ (respectively $J_\sigma = \cap M$), as M runs through all maximal ideals of S for which $f(\sigma, \sigma^{-1}) \notin M$ (respectively $g(\sigma, \sigma^{-1}) \notin M$). We have $J(A_g) \subseteq J(A_f)$ by [8, Theorem 1.5]. 4) By Lemma 2.4, $I_\sigma^{-1} = I_{\sigma^{-1}}$, $J_\sigma^{-1} = J_{\sigma^{-1}}$, and $J_{\sigma^{-1}}g(\sigma^{-1}, \sigma) = J(V)S$.

We conclude, as in the proof of [11, Theorem], that

$$J(V)S \subseteq k_\sigma I_\sigma f(\sigma, \sigma^{-1}). \quad (1)$$

Since $k_\sigma \in U(S_N)$, if $f(\sigma, \sigma^{-1}) \in N^2$ then, localizing both sides of (1) above at N we get $J(S_N) \subseteq J(S_N)^2$, a contradiction, since $J(V)$ is a principal ideal of V . Therefore for each $\tau \in G$ and each maximal ideal M of S , $f(\tau, \tau^{-1}) \notin M^2$. Since the cocycle identity $f^\tau(\tau^{-1}, \tau\gamma)f(\tau, \gamma) = f(\tau, \tau^{-1})$ holds, we conclude that for all $\tau, \gamma \in G$ and every maximal ideal M of S , $f(\tau, \gamma) \notin M^2$.

Conversely, suppose $f(\tau, \gamma) \notin M^2$ for all $\tau, \gamma \in G$, and every maximal ideal M of S . Let $O_l(J(A_f)) = \{x \in \Sigma_f \mid xJ(A_f) \subseteq J(A_f)\}$. We will first establish that $O_l(J(A_f)) = A_f$, again emulating the relevant steps in the proof of [11, Theorem]. To achieve this, it suffices to show that $O_l(J(A_f)) = \sum_{\tau \in G} Sk_\tau x_\tau$ for some $k_\tau \in K^\#$, and that $I_\tau f(\tau, \tau^{-1}) = J(V)S$ for each $\tau \in G$, where I_τ is as in Lemma 2.4. The second assertion follows from

Lemma 2.4(3). As for the first one, we first note that $O_l(J(A_f))$ is a V -order in Σ_f , by [8, Corollary 1.3]. By Lemma 2.5, $O_l(J(A_f)) = \sum_{\tau \in G} Sk_\tau x_\tau$ for some $k_\tau \in K^\#$ if and only if it is finitely generated over V .

Since for all $\tau, \gamma \in G$ and every maximal ideal M of S we have $f(\tau, \gamma) \notin M^2$, we conclude from Lemma 2.3 that $f(G \times G) \subseteq U(R)$, hence B_f is Azumaya over W . Therefore $J(B_f) = J(W)B_f = J(W)(WA_f) = J(W)A_f \subseteq J(A_f)$, and $A_f/J(B_f)$ is a \tilde{V} -order in $B_f/J(B_f)$. Since $O_l(J(A_f))$ is a V -order containing A_f , $O_l(J(A_f))W$ is a W -order containing B_f , so $O_l(J(A_f))W = B_f$, since B_f is a maximal W -order in Σ_f , and hence $O_l(J(A_f)) \subseteq B_f$. Therefore $O_l(J(A_f))/J(B_f)$ is a \tilde{V} -order in $B_f/J(B_f)$, a central simple \overline{W} -algebra. Since \tilde{V} is a DVR of \overline{W} , $O_l(J(A_f))/J(B_f)$ must be finitely generated over \tilde{V} , by [13, Theorem 10.3], hence there exists $a_1, a_2, \dots, a_n \in O_l(J(A_f))$ such that $O_l(J(A_f)) = a_1V + a_2V + \dots + a_nV + J(B_f) = a_1V + a_2V + \dots + a_nV + A_f$, a finitely generated V -module. Thus $O_l(J(A_f)) = A_f$.

As in the proof of [12, Lemma 4.11], we have $O_l(J(A_f/J(B_f))) = O_l(J(A_f)/J(B_f)) = O_l(J(A_f))/J(B_f) = A_f/J(B_f)$, where $O_l(J(A_f/J(B_f)))$ and $O_l(J(A_f)/J(B_f))$ are defined accordingly. Since \tilde{V} is a DVR of \overline{W} , $A_f/J(B_f)$ is a hereditary \tilde{V} -order in the central simple \overline{W} -algebra $B_f/J(B_f)$, hence A_f is semihereditary by [12, Lemma 4.11]. \square

We summarize these results as follows.

Theorem 3.4. *Given a crossed-product order A_f ,*

1. *it is semihereditary if and only if for all $\tau, \gamma \in G$ and every maximal ideal M of S , $f(\tau, \gamma) \notin M^2$; if and only if for each $\gamma \in G$ and each maximal ideal M of S , $f(\tau, \tau^{-1}) \notin M^2$.*
2. *if $J(V)$ is a non-principal ideal of V , then A_f is semihereditary if and only if it is Azumaya over V , if and only if $H = G$.*

We now lump together several corollaries of the theorem above, generalizing results in [11].

Corollary 3.5. 1. *Given a crossed-product order A_f ,*

- (a) *it is a valuation ring if and only if given any maximal ideal M of S , $f(\tau, \tau^{-1}) \notin M^2$ for each $\tau \in G$, and there exists a set of right coset representatives g_1, g_2, \dots, g_r of D_M in G (i.e., G is the disjoint union $\cup_i D_M g_i$) such that for all i , $f(g_i, g_i^{-1}) \notin M$.*

- (b) if V is indecomposed in K , then it is a valuation ring if and only if for each $\tau \in G$, $f(\tau, \tau^{-1}) \notin J(S)^2$.
2. Suppose the crossed-product order A_f is primary. Then it is a valuation ring if and only if there exists a maximal ideal M of S such that for each $\tau \in D_M$, $f(\tau, \tau^{-1}) \notin M^2$.
 3. Suppose the crossed-product order A_f is semihereditary. Then $A_{f_{L,U}}$ is a semihereditary order in $\Sigma_{f_{L,U}}$ for each intermediate field L of F and K , and every valuation ring U of L lying over V .
 4. Suppose the crossed-product order A_f is semihereditary. Then A_{f_M} is a valuation ring of Σ_{f_M} for each maximal ideal M of S .

We end by observing yet another peculiarity of these crossed-product orders. The proposition below not only strengthens Lemma 2.1(2) when the V -order A is taken to be the crossed-product order A_f , but also generalizes [6, Proposition 2.10] to the case where V is not necessarily indecomposed in K .

Proposition 3.6. *Suppose the crossed-product order A_f is extremal and W is a valuation ring of F with $V \subsetneq W$. Then WA_f is Azumaya over W .*

Proof. This follows from Lemma 2.3 and Theorem 3.4. □

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