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ON A CLASS OF HEREDITARY CROSSED-PRODUCT ORDERS

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Dedicated to the memory of my mom

ABSTRACT. In this brief note, we revisit a class of crossed-product orders over discrete valuation rings introduced by D. E. Haile. We give simple but useful criteria, which involve only the two-cocycle associated with a given crossed-product order, for determining whether such an order is a hereditary order or a maximal order.

If R is a ring, then $J(R)$ will denote its Jacobson radical, $U(R)$ its group of multiplicative units, and $R^{\#}$ the subset of all the non-zero elements. The terminology used in this paper, if not in [1], can be found in [3]. The book by Reiner [3] is also an excellent source of literature on maximal orders and hereditary orders.

Let V be a discrete valuation ring (DVR), with quotient field F, and let K/F be a finite Galois extension, with group G, and let S be the integral closure of V in K. Let $f \in Z^2(G, U(K))$ be a normalized two-cocycle. If $f(G \times G) \subseteq S^{\#}$, then one can construct a "crossed-product" V-algebra

$$A_f = \sum_{\sigma \in G} S x_\sigma,$$

with the usual rules of multiplication ($x_\sigma s = \sigma(s) x_\sigma$ for all $s \in S, \sigma \in G$ and $x_\sigma x_\tau = f(\sigma, \tau) x_{\sigma\tau}$). Then A_f is associative, with identity $1 = x_1$, and center $V = V x_1$. Further, A_f is a V-order in the crossed-product F-algebra $\Sigma_f = \sum_{\sigma \in G} K x_\sigma = (K/F, G, f)$.

Two such cocycles f and g are said to be cohomologous over S (respectively cohomologous over K), denoted by $f \sim_S g$ (respectively $f \sim_K g$), if there are elements $\{c_\sigma \mid \sigma \in G\} \subseteq U(S)$ (respectively $\{c_\sigma \mid \sigma \in G\} \subseteq K^{\#}$) such that $g(\sigma, \tau) = c_\sigma \sigma(c_\tau) c_{\sigma\tau}^{-1} f(\sigma, \tau)$ for all $\sigma, \tau \in G$. Following [1], let $H = \{\sigma \in G \mid f(\sigma, \sigma^{-1}) \in U(S)\}$. Then H is a subgroup of G. On G/H, the left coset space of G by H, one can define a partial ordering by the rule $\sigma H \leq \tau H$ if $f(\sigma, \sigma^{-1}\tau) \in U(S)$. Then "$\leq$" is well-defined and depends only on the cohomology class of f over S. Further, H is the unique least element. We call this partial ordering on G/H *the graph of f*.

Such a setup was first formulated by Haile in [1], with the assumption that S is unramified over V, wherein, among other things, conditions equivalent to such orders being maximal orders were considered. This is the class of crossed-product orders we shall study in this paper, *always assuming that S is unramified over V*. We emphasize the fact that, since we do not require that $f(G \times G) \subseteq U(S)$, this

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theory constitutes a drastic departure from the classical theory of crossed-product orders over DVRs, such as can be found in [2].

Let us now fix additional notation to be used in the rest of the paper, most of it borrowed from [1] as before. If M is a maximal ideal of S, let D_M be the decomposition group of M, let K_M be the decomposition field, and let S_M be the localization of S at M. The two-cocycle $f : G \times G \mapsto S^\#$ yields a two-cocycle $f_M : D_M \times D_M \mapsto S_M^\#$, determined by the restriction of f to $D_M \times D_M$ and the inclusion of $S^\#$ in $S_M^\#$. Then $A_{f_M} = \sum_{\sigma \in D_M} S_M x_\sigma$ is a crossed-product order in $\Sigma_{f_M} = \sum_{\sigma \in D_M} K x_\sigma = (K/K_M, D_M, f_M)$. In addition, we can obtain a *twist* of f, described in [1, pp. 137-138] and denoted by $\tilde{f}$, which depends on the choice of a maximal ideal M of S, and the choice of a set of coset representatives of D_M in G. We also define $F : G \times G \mapsto S^\#$ by $F(\sigma, \tau) = f(\sigma, \sigma^{-1}\tau)$ for $\sigma, \tau \in G$. While $\tilde{f}$ is a two-cocycle, F is not.

If B is a V-order of Σ_f containing A_f, then by [1, Proposition 1.3], $B = A_g = \sum_{\sigma \in G} S y_\sigma$ for some two-cocycle $g : G \times G \mapsto S^\#$, with $g \sim_K f$. Moreover, the proof of [1, Proposition 1.3] shows that $y_\sigma = k_\sigma x_\sigma$ for some $k_\sigma \in K^\#$, with $k_1 = 1$, whence g is also a normalized two-cocycle.

We begin with a technical result.

Sublemma. *Let $\tau \in G$. If $I_\tau = \prod_{f(\tau,\tau^{-1}) \notin M} M$, where M denotes a maximal ideal of S, then $I_\tau^{\tau^{-1}} = I_{\tau^{-1}}$.*

Proof. We have

$$I_\tau^{\tau^{-1}} = \prod_{f(\tau,\tau^{-1}) \notin M} M^{\tau^{-1}} = \prod_{f^{\tau^{-1}}(\tau,\tau^{-1}) \notin M^{\tau^{-1}}} M^{\tau^{-1}} = \prod_{f(\tau^{-1},\tau) \notin M^{\tau^{-1}}} M^{\tau^{-1}} = I_{\tau^{-1}}.$$

□

Theorem. *The crossed-product order A_f is hereditary if and only if $f(\tau, \tau^{-1}) \notin M^2$ for all $\tau \in G$ and every maximal ideal M of S.*

Proof. The theorem obviously holds if $H = G$, in which case A_f is an Azumaya algebra over V, so let us assume from now on that $H \neq G$.

Suppose A_f is hereditary. First, assume A_f is a maximal order and S is a DVR. Let v be the valuation corresponding to S with value group $\mathbb{Z}$. Then by [1, Theorem 2.3], H is a normal subgroup of G and G/H is cyclic. Further, there exists $\sigma \in G$ such that $v(f(\sigma, \sigma^{-1})) \leq 1$, $G/H = \langle \sigma H \rangle$, and the graph of f is the chain $H \leq \sigma H \leq \sigma^2 H \leq \cdots \leq \sigma^{m-1} H$, where $m = |G/H|$. Choose j maximal such that $1 \leq j \leq m-1$ and $v(f(\sigma^i, \sigma^{-i})) \leq 1 \ \forall\ 1 \leq i \leq j$. We always have $\sigma H \leq \sigma^{-j} H$; but if $j < m-1$, then we also have $\sigma^j H \leq \sigma^{j+1} H$. Hence if $j < m-1$, then, from the cocycle identity $f^{\sigma^j}(\sigma, \sigma^{-j}\sigma^{-1}) f(\sigma^j, \sigma^{-j}) = f(\sigma^j, \sigma) f(\sigma^{j+1}, \sigma^{-j}\sigma^{-1})$, we conclude that $v(f(\sigma^i, \sigma^{-i})) \leq 1 \ \forall\ 1 \leq i \leq j+1$, a contradiction. So we must have $j = m-1$, so that $v(f(\sigma^i, \sigma^{-i})) \leq 1 \ \forall\ 1 \leq i \leq m-1$. If τ is an arbitrary element of G, then $\tau = \sigma^i h$ for some $h \in H$ and some integer i, $0 \leq i \leq m-1$. Therefore, by [1, Lemma 3.6], $v(f(\tau, \tau^{-1})) = v(F(\sigma^i h, 1)) = v(F(\sigma^i, 1)) = v(f(\sigma^i, \sigma^{-i})) \leq 1$; that is, $f(\tau, \tau^{-1}) \notin J(S)^2$.

We maintain the assumption that A_f is a maximal order, but we now drop the condition that S is a DVR. By [1, Theorem 3.16], there exists a twist of f, say $\tilde{f}$, such that $f \sim_S \tilde{f}$. By [1, Corollary 3.11], for any maximal ideal M of S, A_{f_M} is a maximal order in Σ_{f_M}; hence $f_M(\tau, \tau^{-1}) \notin M^2 \ \forall\ \tau \in D_M$ by the preceding paragraph. Therefore, from the manner in which $\tilde{f}$ is constructed from f, we infer that $\tilde{f}(\tau,\tau^{-1}) \notin M^2\ \forall\ \tau \in G$ and any maximal ideal M of S, and thus $f(\tau,\tau^{-1}) \notin M^2\ \forall\ \tau \in G$ and every maximal ideal M of S, since $f \sim_S \tilde{f}$.

If A_f is not a maximal order, then it is the intersection of finitely many maximal orders, say $A_{f_1}, A_{f_2}, \ldots, A_{f_l}$. Note that

$$A_{f_i} = \sum_{\tau\in G} Sy_\tau^{(i)} = \sum_{\tau\in G} Sk_\tau^{(i)} x_\tau,$$

for some $k_\tau^{(i)} \in K$. Fix a $\sigma \in G$, and a maximal ideal N of S. Let v_N be the valuation corresponding to N, with value group $\mathbb{Z}$. Since

$$S = \bigcap_{i=1}^{l} Sk_\sigma^{(i)},$$

there exists i_0 such that $v_N(k_\sigma^{(i_0)}) = 0$. Let $g = f_{i_0}$ and, for $\tau \in G$, let $k_\tau = k_\tau^{(i_0)}$ and $y_\tau = y_\tau^{(i_0)}$, so that $A_g = \sum_{\tau\in G} Sk_\tau x_\tau = \sum_{\tau\in G} Sy_\tau$. By [1, Proposition 3.1], $J(A_f) = \sum_{\tau\in G} I_\tau x_\tau$ and $J(A_g) = \sum_{\tau\in G} J_\tau y_\tau$, where

$$I_\tau = \prod_{f(\tau,\tau^{-1})\notin M} M \quad\text{and}\quad J_\tau = \prod_{g(\tau,\tau^{-1})\notin M} M,$$

and M denotes a maximal ideal of S. Since A_f is a hereditary V-order in Σ_f and $A_f \subseteq A_g \subseteq \Sigma_f$, we have $J(A_g) \subseteq J(A_f)$, from which we conclude that $J_{\sigma^{-1}} y_{\sigma^{-1}} \subseteq I_{\sigma^{-1}} x_{\sigma^{-1}}$ and so $J_{\sigma^{-1}} k_{\sigma^{-1}} \subseteq I_{\sigma^{-1}}$. We have $y_{\sigma^{-1}} J_\sigma y_\sigma = k_{\sigma^{-1}} x_{\sigma^{-1}} J_\sigma k_\sigma x_\sigma = J_{\sigma^{-1}} k_{\sigma^{-1}} x_{\sigma^{-1}} k_\sigma x_\sigma \subseteq I_{\sigma^{-1}} x_{\sigma^{-1}} k_\sigma x_\sigma = \sigma^{-1}(k_\sigma) I_{\sigma^{-1}} f(\sigma^{-1},\sigma) = \big(k_\sigma I_\sigma f(\sigma,\sigma^{-1})\big)^{\sigma^{-1}}$. On the other hand, $y_{\sigma^{-1}} J_\sigma y_\sigma = J_\sigma^{\sigma^{-1}} g(\sigma^{-1},\sigma) = J_{\sigma^{-1}} g(\sigma^{-1},\sigma)$. Since A_g is a maximal order and therefore $g(\sigma^{-1},\sigma) \notin M^2$ for every maximal ideal M of S, we see that $J_{\sigma^{-1}} g(\sigma^{-1},\sigma) = J(V)S$ and so $y_{\sigma^{-1}} J_\sigma y_\sigma = J(V)S$. Therefore $J(V)S \subseteq k_\sigma I_\sigma f(\sigma,\sigma^{-1})$. Since $v_N(k_\sigma) = 0$, we conclude that $f(\sigma,\sigma^{-1}) \notin N^2$, and so $f(\tau,\tau^{-1}) \notin M^2\ \forall\ \tau \in G$ and any maximal ideal M of S.

Conversely, suppose that $f(\tau,\tau^{-1}) \notin M^2$ for every maximal ideal M of S and every $\tau \in G$. Let $B = O_l(J(A_f))$, the left order of $J(A_f)$; that is, $B = \{x \in \Sigma_f \mid xJ(A_f) \subseteq J(A_f)\}$. Since $\Sigma_f \supseteq B \supseteq A_f$, $B = \sum_{\tau\in G} Sk_\tau x_\tau$, for some $k_\tau \in K^{\#}$. For each $\tau \in G$, we have $S \subseteq Sk_\tau$, and we will now show that $S = Sk_\tau$. As above, write $J(A_f) = \sum I_\tau x_\tau$, with $I_\tau = \prod M$, where the product is taken over all maximal ideals M of S for which $f(\tau,\tau^{-1}) \notin M$. Observe that $J(V)S = I_1 \supseteq k_\tau x_\tau I_{\tau^{-1}} x_{\tau^{-1}} = k_\tau I_{\tau^{-1}}^\tau f(\tau,\tau^{-1}) = k_\tau I_\tau f(\tau,\tau^{-1})$. Since $f(\tau,\tau^{-1}) \notin M^2$ for every maximal ideal M of S, we must have $I_\tau f(\tau,\tau^{-1}) = J(V)S$, and so $J(V)S \supseteq k_\tau J(V)S \supseteq J(V)S$ and thus $S = Sk_\tau$, as desired. This shows that $O_l(J(A_f)) = A_f$ and A_f is hereditary. □

Not only can this criterion enable one to rapidly determine whether or not the crossed-product order A_f is hereditary, the utility of the theorem above is now demonstrated by the ease with which the following corollaries of it are obtained.

Corollary 1. *The crossed-product order A_f is hereditary if and only if $f(\tau,\gamma) \notin M^2$ for all $\tau,\gamma \in G$ and every maximal ideal M of S.*

Proof. This follows from the cocycle identity $f^\tau(\tau^{-1},\tau\gamma)f(\tau,\gamma) = f(\tau,\tau^{-1})$. □

In other words, the order A_f is hereditary if and only if the values of the two-cocycle f are all square-free.

Since A_f is a maximal order if and only if it is hereditary and primary, by combining our result and results in [1], we immediately have the following.

Corollary 2. *Given a crossed-product order A_f,*

(1) *it is a maximal order if and only if for every maximal ideal M of S, $f(\tau, \tau^{-1}) \notin M^2$ for all $\tau \in G$, and there exists a set of right coset representatives $g_1, g_2, \ldots, g_r$ of D_M in G (i.e., G is the disjoint union $\bigcup_i D_M g_i$) such that for all i, $f(g_i, g_i^{-1}) \notin M$.*

(2) *if S is a DVR, then it is a maximal order if and only if $f(\tau, \tau^{-1}) \notin J(S)^2$ for all $\tau \in G$.*

Proof. In either case, the primarity of A_f is guaranteed by [1, Theorem 3.2] (see also [1, Proposition 2.1(b)] when S is a DVR). ☐

The Theorem above can readily be put to effective use with the crossed-product orders in [1, §4], for example. In that section, all the crossed-product orders involved are primary orders, and the two-cocycles are given in tabular form, with the values factorized into primes of S. Using our criterion, it now becomes a straightforward process to determine which of those orders are maximal orders and which are not, by simply consulting, in each case, the given table of values for the two-cocycle; the table whose entries are all square-free represents a maximal order. This determination can be made with little effort! In fact, if one knows that the crossed-product order A_f is a primary order, then determining whether or not it is a maximal order could even be easier, as the following result shows.

Corollary 3. *Suppose the crossed-product order A_f is primary. Then it is a maximal order if and only if there exists a maximal ideal M of S such that $f(\tau, \tau^{-1}) \notin M^2$ for all $\tau \in D_M$.*

Proof. This follows from [1, Corollary 3.11 and Proposition 2.1(b)]. ☐

Let L be an intermediate field of F and K, let G_L be the Galois group of K over L, let U be a valuation ring of L lying over V, and let T be the integral closure of U in K. Then one can obtain a two-cocycle $f_{L,U} : G_L \times G_L \mapsto T^{\#}$ from f by restricting f to $G_L \times G_L$ and embedding $S^{\#}$ in $T^{\#}$. As before, $A_{f_{L,U}} = \sum_{\tau \in G_L} T x_\tau$ is a U-order in $\Sigma_{f_{L,U}} = \sum_{\tau \in G_L} K x_\tau = (K/L, G_L, f_{L,U})$.

Corollary 4. *Suppose the crossed-product order A_f is hereditary. Then $A_{f_{L,U}}$ is a hereditary order in $\Sigma_{f_{L,U}}$ for each intermediate field L of F and K and for every valuation ring U of L lying over V.*

This leads to the following.

Corollary 5. *Suppose the crossed-product order A_f is hereditary. Then A_{f_M} is a maximal order in Σ_{f_M} for each maximal ideal M of S.*

Proof. The order A_{f_M} is always primary, by [1, Proposition 2.1(b)]. ☐

The following example illustrates two limitations of our theory, however.

Example. We give two crossed-product orders A_{f_1} and A_{f_2} with $f_1 \sim_K f_2$ and the graphs of f_1 and f_2 identical, but A_{f_1} is hereditary while A_{f_2} is not. Also, we give an example to demonstrate that the converse of Corollary 5 does not always hold.

Let $F = \mathbb{Q}(x)$, and let $K = \mathbb{Q}(i)(x)$. Then the Galois group $G = \langle\sigma\rangle$ is a group of order two, where σ is induced by the complex conjugation on $\mathbb{Q}(i)$. If $V = \mathbb{Q}[x]_{(x^2+1)}$, then S has two maximal ideals, namely $M_1 = (x+i)S$ and $M_2 = (x-i)S$, and $D_{M_1} = D_{M_2} = \{1\}$. Let $f_1, f_2 : G \times G \mapsto S^{\#}$ be two-cocycles defined by $f_j(1,1) = f_j(1,\sigma) = f_j(\sigma,1) = 1$ and $f_1(\sigma,\sigma) = (x^2+1)x$, $f_2(\sigma,\sigma) = (x^2+1)^2x$.

Then $f_1 \sim_K f_2$, and the subgroup of G associated with either cocycle is $H = \{1\}$, so that the graphs of f_1 and f_2 are identical. Clearly, A_{f_1} is hereditary but A_{f_2} is not. We conclude that the property that a crossed-product order A_f is hereditary is not an intrinsic property of the graph of f.

Also, if we set $f = f_2$, we see that $A_{f_M} = S_M$ for each maximal ideal M of S, and therefore A_{f_M} is a maximal order in $\Sigma_{f_M} = K$ for each maximal ideal M of S, and yet A_f is not even hereditary (cf. [1, Corollary 3.11] and [2, Theorem 1]). This is the case because A_f is not primary, and also because $f(G \times G) \not\subseteq U(S)$. □

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