

# On a class of hereditary crossed-product orders

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## Abstract

In this brief note, we revisit a class of crossed-product orders over discrete valuation rings introduced by D. E. Haile. We give simple but useful criteria, which involve only the two-cocycle associated with a given crossed-product order, for determining whether such an order is a hereditary order or a maximal order.

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If  $R$  is a ring, then  $J(R)$  will denote its Jacobson radical,  $U(R)$  its group of multiplicative units, and  $R^\#$  the subset of all the non-zero elements. The terminology used in this paper, if not in [1], can be found in [3]. The book by Reiner [3] is also an excellent source of literature on maximal orders and hereditary orders.

Let  $V$  be a discrete valuation ring (DVR), with quotient field  $F$ , and let  $K/F$  be a finite Galois extension, with group  $G$ , and let  $S$  be the integral closure of  $V$  in  $K$ . Let  $f \in Z^2(G, U(K))$  be a normalized two-cocycle. If  $f(G \times G) \subseteq S^\#$ , then one can construct a “crossed-product”  $V$ -algebra

$$A_f = \sum_{\sigma \in G} Sx_\sigma,$$

with the usual rules of multiplication ( $x_\sigma s = \sigma(s)x_\sigma$  for all  $s \in S, \sigma \in G$  and  $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$ ). Then  $A_f$  is associative, with identity  $1 = x_1$ , and center  $V = Vx_1$ . Further,  $A_f$  is a  $V$ -order in the crossed-product  $F$ -algebra  $\Sigma_f = \sum_{\sigma \in G} Kx_\sigma = (K/F, G, f)$ .

Two such cocycles  $f$  and  $g$  are said to be cohomologous over  $S$  (respectively cohomologous over  $K$ ), denoted by  $f \sim_S g$  (respectively  $f \sim_K g$ ), if there are elements  $\{c_\sigma \mid \sigma \in G\} \subseteq U(S)$  (respectively  $\{c_\sigma \mid \sigma \in G\} \subseteq K^\#$ ) such that  $g(\sigma, \tau) = c_\sigma \sigma(c_\tau) c_{\sigma\tau}^{-1} f(\sigma, \tau)$  for all  $\sigma, \tau \in G$ . Following [1], let  $H = \{\sigma \in G \mid f(\sigma, \sigma^{-1}) \in U(S)\}$ . Then  $H$  is a subgroup of  $G$ . On  $G/H$ , the left coset space of  $G$  by  $H$ , one can define a partial ordering by the rule  $\sigma H \leq \tau H$  if  $f(\sigma, \sigma^{-1}\tau) \in U(S)$ . Then “ $\leq$ ” is well-defined and depends only on the cohomology class of  $f$  over  $S$ . Further,  $H$  is the unique least element. We call this partial ordering on  $G/H$  the *graph of  $f$* .

Such a setup was first formulated by Haile in [1], with the assumption that  $S$  is unramified over  $V$ , wherein, among other things, conditions equivalent to such orders being maximal orders were considered. This is the class of crossed-product orders we shall study in this paper, *always assuming that  $S$  is unramified*

over  $V$ . We emphasize the fact that, since we do not require that  $f(G \times G) \subseteq U(S)$ , this theory constitutes a drastic departure from the classical theory of crossed-product orders over DVRs, such as can be found in [2].

Let us now fix additional notation to be used in the rest of the paper, most of it borrowed from [1] as before. If  $M$  is a maximal ideal of  $S$ , let  $D_M$  be the decomposition group of  $M$ , let  $K_M$  be the decomposition field, and let  $S_M$  be the localization of  $S$  at  $M$ . The two-cocycle  $f : G \times G \mapsto S^\#$  yields a two-cocycle  $f_M : D_M \times D_M \mapsto S_M^\#$ , determined by the restriction of  $f$  to  $D_M \times D_M$  and the inclusion of  $S^\#$  in  $S_M^\#$ . Then  $A_{f_M} = \sum_{\sigma \in D_M} S_M x_\sigma$  is a crossed-product order in  $\Sigma_{f_M} = \sum_{\sigma \in D_M} K x_\sigma = (K/K_M, D_M, f_M)$ . In addition, we can obtain a *twist* of  $f$ , described in [1, pp. 137-138] and denoted by  $\tilde{f}$ , which depends on the choice of a maximal ideal  $M$  of  $S$ , and the choice of a set of coset representatives of  $D_M$  in  $G$ . We also define  $F : G \times G \mapsto S^\#$  by  $F(\sigma, \tau) = f(\sigma, \sigma^{-1}\tau)$  for  $\sigma, \tau \in G$ . While  $\tilde{f}$  is a two-cocycle,  $F$  is not.

If  $B$  is a  $V$ -order of  $\Sigma_f$  containing  $A_f$ , then by [1, Proposition 1.3],  $B = A_g = \sum_{\sigma \in G} S y_\sigma$  for some two-cocycle  $g : G \times G \mapsto S^\#$ , with  $g \sim_K f$ . Moreover, the proof of [1, Proposition 1.3] shows that  $y_\sigma = k_\sigma x_\sigma$  for some  $k_\sigma \in K^\#$ , with  $k_1 = 1$ , whence  $g$  is also a normalized two-cocycle.

We begin with a technical result.

**Sublemma.** *Let  $\tau \in G$ . If  $I_\tau = \prod_{f(\tau, \tau^{-1}) \notin M} M$ , where  $M$  denotes a maximal ideal of*

*$S$ , then  $I_\tau^{\tau^{-1}} = I_{\tau^{-1}}$ .*

*Proof.* We have

$$I_\tau^{\tau^{-1}} = \prod_{f(\tau, \tau^{-1}) \notin M} M^{\tau^{-1}} = \prod_{f^{\tau^{-1}}(\tau, \tau^{-1}) \notin M^{\tau^{-1}}} M^{\tau^{-1}} = \prod_{f(\tau^{-1}, \tau) \notin M^{\tau^{-1}}} M^{\tau^{-1}} = I_{\tau^{-1}}.$$

□

**Theorem.** *The crossed-product order  $A_f$  is hereditary if and only if  $f(\tau, \tau^{-1}) \notin M^2$  for all  $\tau \in G$  and every maximal ideal  $M$  of  $S$ .*

*Proof.* The theorem obviously holds if  $H = G$ , in which case  $A_f$  is an Azumaya algebra over  $V$ , so let us assume from now on that  $H \neq G$ .

Suppose  $A_f$  is hereditary. First, assume  $A_f$  is a maximal order and  $S$  is a DVR. Let  $v$  be the valuation corresponding to  $S$  with value group  $\mathbb{Z}$ . Then by [1, Theorem 2.3],  $H$  is a normal subgroup of  $G$  and  $G/H$  is cyclic. Further, there exists  $\sigma \in G$  such that  $v(f(\sigma, \sigma^{-1})) \leq 1$ ,  $G/H = \langle \sigma H \rangle$ , and the graph of  $f$  is the chain  $H \leq \sigma H \leq \sigma^2 H \leq \dots \leq \sigma^{m-1} H$ , where  $m = |G/H|$ . Choose  $j$  maximal such that  $1 \leq j \leq m-1$  and  $v(f(\sigma^i, \sigma^{-i})) \leq 1 \forall 1 \leq i \leq j$ . We always have  $\sigma H \leq \sigma^{-j} H$ ; but if  $j < m-1$ , then we also have  $\sigma^j H \leq \sigma^{j+1} H$ . Hence if  $j < m-1$ , then, from the cocycle identity  $f^{\sigma^j}(\sigma, \sigma^{-j}\sigma^{-1})f(\sigma^j, \sigma^{-j}) = f(\sigma^j, \sigma)f(\sigma^{j+1}, \sigma^{-j}\sigma^{-1})$ , we conclude that  $v(f(\sigma^i, \sigma^{-i})) \leq 1 \forall 1 \leq i \leq j+1$ , a contradiction. So we must have  $j = m-1$ , so that  $v(f(\sigma^i, \sigma^{-i})) \leq 1 \forall 1 \leq i \leq m-1$ . If  $\tau$  is an arbitrary element of  $G$ , then  $\tau = \sigma^i h$  for some  $h \in H$  and

some integer  $i$ ,  $0 \leq i \leq m-1$ . Therefore, by [1, Lemma 3.6],  $v(f(\tau, \tau^{-1})) = v(F(\sigma^i h, 1)) = v(F(\sigma^i, 1)) = v(f(\sigma^i, \sigma^{-i})) \leq 1$ ; that is,  $f(\tau, \tau^{-1}) \notin J(S)^2$ .

We maintain the assumption that  $A_f$  is a maximal order, but we now drop the condition that  $S$  is a DVR. By [1, Theorem 3.16], there exists a twist of  $f$ , say  $\tilde{f}$ , such that  $f \sim_S \tilde{f}$ . By [1, Corollary 3.11], for any maximal ideal  $M$  of  $S$ ,  $A_{f_M}$  is a maximal order in  $\Sigma_{f_M}$ ; hence  $f_M(\tau, \tau^{-1}) \notin M^2 \forall \tau \in D_M$  by the preceding paragraph. Therefore, from the manner in which  $\tilde{f}$  is constructed from  $f$ , we infer that  $\tilde{f}(\tau, \tau^{-1}) \notin M^2 \forall \tau \in G$  and any maximal ideal  $M$  of  $S$ , and thus  $f(\tau, \tau^{-1}) \notin M^2 \forall \tau \in G$  and every maximal ideal  $M$  of  $S$ , since  $f \sim_S \tilde{f}$ .

If  $A_f$  is not a maximal order, then it is the intersection of finitely many maximal orders, say  $A_{f_1}, A_{f_2}, \dots, A_{f_l}$ . Note that

$$A_{f_i} = \sum_{\tau \in G} S y_\tau^{(i)} = \sum_{\tau \in G} S k_\tau^{(i)} x_\tau,$$

for some  $k_\tau^{(i)} \in K$ . Fix a  $\sigma \in G$ , and a maximal ideal  $N$  of  $S$ . Let  $v_N$  be the valuation corresponding to  $N$ , with value group  $\mathbb{Z}$ . Since

$$S = \bigcap_{i=1}^l S k_\sigma^{(i)},$$

there exists  $i_0$  such that  $v_N(k_\sigma^{(i_0)}) = 0$ . Let  $g = f_{i_0}$  and, for  $\tau \in G$ , let  $k_\tau = k_\tau^{(i_0)}$  and  $y_\tau = y_\tau^{(i_0)}$ , so that  $A_g = \sum_{\tau \in G} S k_\tau x_\tau = \sum_{\tau \in G} S y_\tau$ . By [1, Proposition 3.1],  $J(A_f) = \sum_{\tau \in G} I_\tau x_\tau$  and  $J(A_g) = \sum_{\tau \in G} J_\tau y_\tau$ , where

$$I_\tau = \prod_{f(\tau, \tau^{-1}) \notin M} M \quad \text{and} \quad J_\tau = \prod_{g(\tau, \tau^{-1}) \notin M} M,$$

and  $M$  denotes a maximal ideal of  $S$ . Since  $A_f$  is a hereditary  $V$ -order in  $\Sigma_f$  and  $A_f \subseteq A_g \subseteq \Sigma_f$ , we have  $J(A_g) \subseteq J(A_f)$ , from which we conclude that  $J_{\sigma^{-1}} y_{\sigma^{-1}} \subseteq I_{\sigma^{-1}} x_{\sigma^{-1}}$  and so  $J_{\sigma^{-1}} k_{\sigma^{-1}} \subseteq I_{\sigma^{-1}}$ . We have  $y_{\sigma^{-1}} J_\sigma y_\sigma = k_{\sigma^{-1}} x_{\sigma^{-1}} J_\sigma k_\sigma x_\sigma = J_{\sigma^{-1}} k_{\sigma^{-1}} x_{\sigma^{-1}} k_\sigma x_\sigma \subseteq I_{\sigma^{-1}} x_{\sigma^{-1}} k_\sigma x_\sigma = \sigma^{-1}(k_\sigma) I_{\sigma^{-1}} f(\sigma^{-1}, \sigma) = (k_\sigma I_\sigma f(\sigma, \sigma^{-1}))^{\sigma^{-1}}$ . On the other hand,  $y_{\sigma^{-1}} J_\sigma y_\sigma = J_\sigma^{\sigma^{-1}} g(\sigma^{-1}, \sigma) = J_{\sigma^{-1}} g(\sigma^{-1}, \sigma)$ . Since  $A_g$  is a maximal order and therefore  $g(\sigma^{-1}, \sigma) \notin M^2$  for every maximal ideal  $M$  of  $S$ , we see that  $J_{\sigma^{-1}} g(\sigma^{-1}, \sigma) = J(V)S$  and so  $y_{\sigma^{-1}} J_\sigma y_\sigma = J(V)S$ . Therefore  $J(V)S \subseteq k_\sigma I_\sigma f(\sigma, \sigma^{-1})$ . Since  $v_N(k_\sigma) = 0$ , we conclude that  $f(\sigma, \sigma^{-1}) \notin N^2$ , and so  $f(\tau, \tau^{-1}) \notin M^2 \forall \tau \in G$  and any maximal ideal  $M$  of  $S$ .

Conversely, suppose that  $f(\tau, \tau^{-1}) \notin M^2$  for every maximal ideal  $M$  of  $S$  and every  $\tau \in G$ . Let  $B = O_l(J(A_f))$ , the left order of  $J(A_f)$ ; that is,  $B = \{x \in \Sigma_f \mid xJ(A_f) \subseteq J(A_f)\}$ . Since  $\Sigma_f \supseteq B \supseteq A_f$ ,  $B = \sum_{\tau \in G} S k_\tau x_\tau$ , for some  $k_\tau \in K^\#$ . For each  $\tau \in G$ , we have  $S \subseteq S k_\tau$ , and we will now show that  $S = S k_\tau$ . As above, write  $J(A_f) = \sum I_\tau x_\tau$ , with  $I_\tau = \prod M$ , where the product is taken over all maximal ideals  $M$  of  $S$  for which  $f(\tau, \tau^{-1}) \notin M$ . Observe

that  $J(V)S = I_1 \supseteq k_\tau x_\tau I_{\tau^{-1}} x_{\tau^{-1}} = k_\tau I_{\tau^{-1}}^\tau f(\tau, \tau^{-1}) = k_\tau I_\tau f(\tau, \tau^{-1})$ . Since  $f(\tau, \tau^{-1}) \notin M^2$  for every maximal ideal  $M$  of  $S$ , we must have  $I_\tau f(\tau, \tau^{-1}) = J(V)S$ , and so  $J(V)S \supseteq k_\tau J(V)S \supseteq J(V)S$  and thus  $S = Sk_\tau$ , as desired. This shows that  $O_i(J(A_f)) = A_f$  and  $A_f$  is hereditary.  $\square$

Not only can this criterion enable one to rapidly determine whether or not the crossed-product order  $A_f$  is hereditary, the utility of the theorem above is now demonstrated by the ease with which the following corollaries of it are obtained.

**Corollary 1.** *The crossed-product order  $A_f$  is hereditary if and only if  $f(\tau, \gamma) \notin M^2$  for all  $\tau, \gamma \in G$  and every maximal ideal  $M$  of  $S$ .*

*Proof.* This follows from the cocycle identity  $f^\tau(\tau^{-1}, \tau\gamma)f(\tau, \gamma) = f(\tau, \tau^{-1})$ .  $\square$

In other words, the order  $A_f$  is hereditary if and only if the values of the two-cocycle  $f$  are all square-free.

Since  $A_f$  is a maximal order if and only if it is hereditary and primary, by combining our result and results in [1], we immediately have the following.

**Corollary 2.** *Given a crossed-product order  $A_f$ ,*

1. *it is a maximal order if and only if for every maximal ideal  $M$  of  $S$ ,  $f(\tau, \tau^{-1}) \notin M^2$  for all  $\tau \in G$ , and there exists a set of right coset representatives  $g_1, g_2, \dots, g_r$  of  $D_M$  in  $G$  (i.e.,  $G$  is the disjoint union  $\cup_i D_M g_i$ ) such that for all  $i$ ,  $f(g_i, g_i^{-1}) \notin M$ .*
2. *if  $S$  is a DVR, then it is a maximal order if and only if  $f(\tau, \tau^{-1}) \notin J(S)^2$  for all  $\tau \in G$ .*

*Proof.* In either case, the primarity of  $A_f$  is guaranteed by [1, Theorem 3.2] (see also [1, Proposition 2.1(b)] when  $S$  is a DVR).  $\square$

The Theorem above can readily be put to effective use with the crossed-product orders in [1, §4], for example. In that section, all the crossed-product orders involved are primary orders, and the two-cocycles are given in tabular form, with the values factorized into primes of  $S$ . Using our criterion, it now becomes a straightforward process to determine which of those orders are maximal orders and which are not, by simply consulting, in each case, the given table of values for the two-cocycle; the table whose entries are all square-free represents a maximal order. This determination can be made with little effort! In fact, if one knows that the crossed-product order  $A_f$  is a primary order, then determining whether or not it is a maximal order could even be easier, as the following result shows.

**Corollary 3.** *Suppose the crossed-product order  $A_f$  is primary. Then it is a maximal order if and only if there exists a maximal ideal  $M$  of  $S$  such that  $f(\tau, \tau^{-1}) \notin M^2$  for all  $\tau \in D_M$ .*

*Proof.* This follows from [1, Corollary 3.11 and Proposition 2.1(b)].  $\square$

Let  $L$  be an intermediate field of  $F$  and  $K$ , let  $G_L$  be the Galois group of  $K$  over  $L$ , let  $U$  be a valuation ring of  $L$  lying over  $V$ , and let  $T$  be the integral closure of  $U$  in  $K$ . Then one can obtain a two-cocycle  $f_{L,U} : G_L \times G_L \mapsto T^\#$  from  $f$  by restricting  $f$  to  $G_L \times G_L$  and embedding  $S^\#$  in  $T^\#$ . As before,  $A_{f_{L,U}} = \sum_{\tau \in G_L} Tx_\tau$  is a  $U$ -order in  $\Sigma_{f_{L,U}} = \sum_{\tau \in G_L} Kx_\tau = (K/L, G_L, f_{L,U})$ .

**Corollary 4.** *Suppose the crossed-product order  $A_f$  is hereditary. Then  $A_{f_{L,U}}$  is a hereditary order in  $\Sigma_{f_{L,U}}$  for each intermediate field  $L$  of  $F$  and  $K$  and for every valuation ring  $U$  of  $L$  lying over  $V$ .*

This leads to the following.

**Corollary 5.** *Suppose the crossed-product order  $A_f$  is hereditary. Then  $A_{f_M}$  is a maximal order in  $\Sigma_{f_M}$  for each maximal ideal  $M$  of  $S$ .*

*Proof.* The order  $A_{f_M}$  is always primary, by [1, Proposition 2.1(b)].  $\square$

The following example illustrates two limitations of our theory, however.

**Example.** We give two crossed-product orders  $A_{f_1}$  and  $A_{f_2}$  with  $f_1 \sim_K f_2$  and the graphs of  $f_1$  and  $f_2$  identical, but  $A_{f_1}$  is hereditary while  $A_{f_2}$  is not. Also, we give an example to demonstrate that the converse of Corollary 5 does not always hold.

Let  $F = \mathbb{Q}(x)$ , and let  $K = \mathbb{Q}(i)(x)$ . Then the Galois group  $G = \langle \sigma \rangle$  is a group of order two, where  $\sigma$  is induced by the complex conjugation on  $\mathbb{Q}(i)$ . If  $V = \mathbb{Q}[x]_{(x^2+1)}$ , then  $S$  has two maximal ideals, namely  $M_1 = (x+i)S$  and  $M_2 = (x-i)S$ , and  $D_{M_1} = D_{M_2} = \{1\}$ . Let  $f_1, f_2 : G \times G \mapsto S^\#$  be two-cocycles defined by  $f_j(1, 1) = f_j(1, \sigma) = f_j(\sigma, 1) = 1$  and  $f_1(\sigma, \sigma) = (x^2 + 1)x$ ,  $f_2(\sigma, \sigma) = (x^2 + 1)^2x$ .

Then  $f_1 \sim_K f_2$ , and the subgroup of  $G$  associated with either cocycle is  $H = \{1\}$ , so that the graphs of  $f_1$  and  $f_2$  are identical. Clearly,  $A_{f_1}$  is hereditary but  $A_{f_2}$  is not. We conclude that the property that a crossed-product order  $A_f$  is hereditary is not an intrinsic property of the graph of  $f$ .

Also, if we set  $f = f_2$ , we see that  $A_{f_M} = S_M$  for each maximal ideal  $M$  of  $S$ , and therefore  $A_{f_M}$  is a maximal order in  $\Sigma_{f_M} = K$  for each maximal ideal  $M$  of  $S$ , and yet  $A_f$  is not even hereditary (cf. [1, Corollary 3.11], and [2, Theorem 1]). This is the case because  $A_f$  is not primary, and also because  $f(G \times G) \not\subseteq U(S)$ .  $\square$

## References

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