Real Valuations on Skew Polynomial Rings

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Let $R = D[T; \sigma, \delta]$ be a left skew polynomial ring in T over a division ring D. For a fixed proper real valuation ν on D that satisfies the condition $\nu(\sigma(a)) = \nu(a)$ for each $a \in D$, let $Val_\nu(R)$ denote the set of all real valuations μ on R extending ν. (The proof of the existence of such extensions can be found, among other places, in [R. C. Churchill and Y. Zhang, J. Algebra 322 (2009), no. 11, 3797–3822. MR2556124 (2010m:16040)].) Define a partial ordering $\preceq$ on $Val_\nu(R)$ by $\mu \preceq \widetilde{\mu}$ when $\mu(f) \leq \widetilde{\mu}(f)$ for all $f \in R$. Through an iterative method, the authors construct a strictly increasing map $a_\nu : Val_\nu(R) \mapsto \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with remarkable properties that will be described below. The main result of the paper is that the poset $(Val_\nu(R), \preceq)$ satisfies the following conditions:

(a) **It is a parameterized non-metric tree:** for (1) given $\mu, \widetilde{\mu} \in Val_\nu(R)$, there exists $\mu_* \in Val_\nu(R)$ such that $\mu_* \preceq \mu$ and $\mu_* \preceq \widetilde{\mu}$;(2) if $\mathcal{J}_\mu = \{\mu' \in Val_\nu(R) : \mu' \preceq \mu\}$, then $a_\nu(\mathcal{J}_\mu)$ is an interval in $\overline{\mathbb{R}}$; and (3) if $\mathcal{S}$ is a full, totally ordered subset of $Val_\nu(R)$ (i.e., if $\mathcal{S}$ satisfies the property that whenever $\mu, \widetilde{\mu} \in \mathcal{S}$ with $\mu \preceq \widetilde{\mu}$ then $[\mu, \widetilde{\mu}] \subseteq \mathcal{S}$), then $a_\nu(\mathcal{S})$ is again an interval in $\overline{\mathbb{R}}$. This shows that $(Val_\nu(R), \preceq)$ is indeed a non-metric tree as defined in [Favre, Charles; Jonsson, Mattias. *The valuative tree*, Lecture Notes in Mathematics, 1853. Springer-Verlag, Berlin, 2004. MR2097722 (2006a:13008)], and in addition, condition (3) is also saying that this non-metric tree is "parameterized" by the map a_ν.

(b) **It is a complete non-metric tree:** for given an increasing sequence μ_i in $Val_\nu(R)$, it is bounded above (e.g. by $\widetilde{\mu} \in Val_\nu(R)$ given by $\widetilde{\mu}(f) = \sup_i \mu_i(f)$).

This generalizes a result by V. G. Berkovich [*Spectral Theory and Analytic Geometry over non-Archimedean Fields*, Mathematical Surveys and Monographs,

33. American Mathematical Society, Providence, RI, 1990. MR1070709 (91k:32038)]. As an application, the authors produce an irreducibility criterion for left skew polynomial rings that generalizes the version of the Eisenstein Criterion by R. C. Churchill and Y. Zhang [op. cit.].