# Monomial, Gorenstein and Bass orders

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Let $K$ be a non-archimedean local field with ring of integers $\mathcal{O}$, let $D$ be a central division algebra over $K$, let $\mathcal{O}_D$ be the unique maximal $\mathcal{O}$-order in $D$, and let $\mathcal{B}$ be the maximal ideal of $\mathcal{O}_D$. The authors define a *monomial* $\mathcal{O}$-order $R$ in $A := M_n(D)$ as an $\mathcal{O}$-order which is isomorphic to one in block matrix form: $R = (\mathcal{B}^{m_{ij}})$, where $m = (m_{ij})$ is an integer $n \times n$ matrix which is referred to as the *level* of $R$. In other words, it is an $\mathcal{O}$-order which is also a $\mathcal{O}_D$-submodule of $A$ and contains a full set of primitive orthogonal idempotents of $A$. Therefore the class of monomial orders has hereditary orders as a subclass. Further, any monomial order is conjugate to one that contains the standard idempotents of $A$, which in turn is exactly the type of an $\mathcal{O}$-order of $A$ that can be expressed in block matrix form as above.

A monomial order $R$ that contains the standard idempotents of $A$ is referred to as a *standard monomial order* in this paper. A standard monomial order is said to be *upper triangular* if $m$ is a strict lower triangular matrix. It is shown that any standard monomial order is conjugate to one whereby $m_{ij} \geq 0$ for all $i, j$, and it is assumed throughout the paper that the monomial order $R$ is of this form. The authors define an *Eichler order* in $A$ as an upper triangular monomial order with the property that for some fixed positive integer $a$, we have $m_{ij} \in \{0, a\}$ for all $i > j$. This agrees with the usual definition when $A$ is a quaternion algebra over $K$.

The authors characterize monomial orders that are Gorenstein orders and use this characterization to demonstrate that, if $R$ is upper triangular, then it is Gorenstein precisely when $R$ is an Eichler order. Again assuming that $R$ is upper triangular, they show that $R$ is a Bass order precisely when it is either a hereditary order, or an Eichler order with

$$m = \begin{bmatrix} \mathbf{0}_{k_1}^{k_1} & \mathbf{0}_{k_2}^{k^1} \\ a\mathbf{1}_{k_1}^{k_2} & \mathbf{0}_{k_2}^{k_2} \end{bmatrix},$$

where $k_i, a$ are positive integers with $k_1 + k_2 = n$ and $a\mathbf{1}_s^r$ (resp. $\mathbf{0}_s^r$) are

$r \times s$ matrices with every entry equal to $a$ (resp. 0). All this lends credence to the assertion made in the paper that monomial orders can be regarded as generalizations to central simple algebras of Eichler orders in quaternion algebras. For definitions of Gorenstein orders and Bass orders, see [C. W. Curtis and I. Reiner, *Methods of representation theory. Vol. I*, Wiley, New York, 1981; MR0632548 (82i:20001)]

A few comments are in order. There are some errors in the proof of Theorem 3.1: the matrix $E_{ij}$ has been mislabelled as an elementary matrix, which it is not; matrix Equation (3.7) appears to be malformed; and there is a typo in the last sentence of the proof. All these add up to make the validation of the theorem rather difficult. Otherwise there are many original computational techniques in this paper which are quite valuable in and of themselves.