

THE REALS AS RATIONAL CAUCHY FILTERS

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Abstract. We present, alongside a historical note on the development of the study of the real numbers, a detailed and elementary construction of the real numbers from the rational numbers a la Bourbaki. The real numbers are defined to be the set of all minimal Cauchy filters in \mathbb{Q} (where the Cauchy condition is defined in terms of the absolute value function on \mathbb{Q}) and are proven directly, without employing any of the techniques of uniform spaces, to form a complete ordered field. The construction can be seen as a variant of Bachmann's construction by means of nested rational intervals, allowing for a canonical choice of representatives.

1. Introduction

The aim of this article is to present, together with a short historical survey of the early developments of our understanding of the real numbers, by means of a tour of several pivotal achievements, a construction of the real numbers. Of the numerous ways of constructing the real numbers (see [22] for a survey, where the present construction is outlined, christened the Bourbaki reals) perhaps the two most famous approaches are Cantor's and Dedekind's. For reasons explained below, we propose the present construction as a competitor in the categories of elegance and of pedagogical importance to these two constructions.

For the convenience of the reader the article is designed to flexibly accommodate itself for readers who may have different aims when approaching it. The reader wishing to immediately see the new construction of the real numbers may directly proceed to read Section 4, referring, if needed to the preliminaries in Section 3. The reader only interested in a brief motivation for the new construction prior to delving into its details will be satisfied with first reading this introduction. The reader also interested in criticism of the common-place constructions by means of Cauchy sequences and by means of Dedekind cuts should first read Subsection 2.2. Lastly, for the reader interested in the full historical perspective offered in Section 2, the progression of ideas presented is largely chronological, so that she may easily navigate to her most preferred era.

The construction of the real numbers we present is quickly motivated in one of two ways. Bourbaki's approach to the real numbers (see [4]) is not to construct any particular model of the real numbers, but rather to view them as a completion of \mathbb{Q} , considered as a uniform space. The proofs of the complete ordered field axioms are facilitated through the use of general universal properties of the completion,

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thus avoiding the technicalities of any particular construction. While the elegance of this approach is undisputed it does not constitute a proof of the consistency of the axioms of complete ordered fields (relative to the rationals) without recourse to the existence of the completion of a uniform space. Of course, Bourbaki also provides a general construction of such a completion in terms of minimal Cauchy filters. Thus, following Bourbaki and at the same time going against Bourbaki's spirit of not constructing the reals, we do construct the reals as minimal Cauchy filters of rational numbers. The aim of this work is to present the details of this construction in a completely elementary and self-contained fashion.

A second motivation for the construction goes back to the late 19th century and the early attempts of placing the real numbers on a rigorous footing. Other than Cantor's construction by means of Cauchy sequences and Dedekind's construction by cuts, some of the other attempts for formal constructions of the real numbers were motivated by the properties of nested intervals, though working out the details proved challenging, finally culminating with Bachmann's construction ([2]). Some of the general interest in the potential of each of these three alternatives is gathered from the following quote from [9, page 46]

The practical advantages of nested intervals over cuts or fundamental sequences are as follows. If the real number x is described by (I_n) the position of x on the number axis is fixed within defined bounds by each I_n . On the other hand with a fundamental sequence (r_n) , the knowledge of one r_n still tells us nothing about the position of x . Again, the description of x as a cut $(\bar{\alpha}, \beta)$ can result from a definition of the set α by means of statements which say nothing directly about the position of x . The theoretical disadvantage of using the nested interval approach is that introducing the relation \leq between equivalence classes of nets of nested intervals and verifying the field properties for addition and multiplication is somewhat troublesome.

Let us now note that any nest $\{I_n\}$ of rational intervals is a Cauchy filter base and thus gives rise to a unique minimal Cauchy filter of rationals (see Subsection 3.2 for details). This observation gives a direct comparison between Bachmann's construction and our construction, and in this sense our construction can be seen as a variant of Bachmann's allowing for a canonical choice of representatives. It is shown below (i.e., Proposition 4.12) that the practical advantages of Bachmann's construction are shared with our construction, while the technical difficulties one encounters with Bachmann's construction are confined in our construction to the proof of one result (i.e., Lemma 4.22). The rest of the construction is rather straightforward. Moreover, unlike in Bachmann's construction, in our construction the definition of the ordering of the reals, and the related proofs, are quite elegant. We thus hope to place Bachmann's construction, through the variant we present, as a competitor of potentially equal popularity as either Cantor's construction or Dedekind's construction.

The real numbers are defined below to be the set of all minimal Cauchy filters in \mathbb{Q} . The ordering on the reals is given as follows. For real numbers a and b , we

declare that

$$a < b$$

precisely when there exist $A \in a$ and $B \in b$ such that

$$A < B$$

universally, i.e., when

$$\alpha < \beta$$

for all $\alpha \in A$ and $\beta \in B$. Equivalently, $a \leq b$ holds precisely when for all $A \in a$ and $B \in b$

$$A \leq B$$

existentially, i.e., when

$$\alpha \leq \beta$$

for some $\alpha \in A$ and some $\beta \in B$. The algebraic structure on \mathbb{R} is defined as follows. Given real numbers a and b , their *sum*

$$a + b$$

is the filter generated by the filter base

$$\{A + B \mid A \in a, B \in b\},$$

where

$$A + B = \{\alpha + \beta \mid \alpha \in A, \beta \in B\}.$$

Similarly, the *product*

$$ab$$

is the filter generated by the filter base

$$\{AB \mid A \in a, B \in b\},$$

where

$$AB = \{\alpha\beta \mid \alpha \in A, \beta \in B\}.$$

Below we give a detailed proof that the reals thus defined form a complete ordered field, without a-priori use of uniform structures or the completion process by means of minimal Cauchy filters. Thus the treatment is completely elementary.

Remark 1.1. *A word on the originality content of this work is in order. Bourbaki's construction of the reals as the uniform completion of the rationals is certainly not new, nor is the use of minimal Cauchy filters in the construction of the completion of any uniform space. Due to the ambient well-developed general theory we find ourselves in the position of being guaranteed that constructing the reals as minimal Cauchy rational filters must work. However, the details of this construction as we present below, other than being elementary, are not just the result of unpacking the classical Bourbaki proofs. The definitions of addition and multiplication of real numbers are given explicitly on the level of the minimal Cauchy filters without the use of the roundification process of a filter. The ordering structure, which Bourbaki gives in terms of differences and positives, is also given directly in terms of the minimal Cauchy filters, and this is perhaps the main contribution of this work in terms of originality.*

1.1. Plan of the paper. The construction of the reals is given in section §4 where we take as given a model \mathbb{Q} for the rationals as an ordered archimedean field. Since the prerequisites for the construction are very modest, section §3 is a self-contained preliminary section giving an elementary treatment of the geometry of intervals and the basics of filters. After the construction is dealt with, section §5 is a short presentation of two consequences of the formalism - a proof of the uncountability of the reals and a criterion for convergence. For the sake of presenting the reader with a broader spectrum than just the details of the construction, section §2 is a journey to some of the realms of modern mathematics inspired by the real numbers.

2. The Real Numbers and their Role in Shaping Mathematics

In this section we provide a brief account of the real numbers, of how the real numbers are typically modeled, and of the reciprocal effects between the ambient mathematics and the desire to deepen our understanding of the real numbers. Since a full treatment of these issues can easily fill up an entire book, the material presented is by necessity partial.

2.1. Prehistory. The ancient Greeks, and in particular the Pythagoreans, were very fond of numbers. Alongside laying the foundations of modern axiomatic rigor they also held magnificent superstitious beliefs about how numbers relate to, and govern, nearly everything in the universe. The Greeks' concept of a number was slightly different than our modern understanding of what it is. While we accept number systems even if they are not used (immediately) to measure anything real, the Pythagoreans were highly motivated by geometry, and numbers were often used for, and understood through, geometric interpretation. For many years the Greek mathematicians and engineers were quite satisfied with their system of ratios - a system quite close to the modern system of rational numbers. The common belief was that ratios suffice for all practical real needs, or that at the very least they suffice to measure all geometric constructs precisely. It is thus that the famous discovery (about which very few details are known) that $\sqrt{2}$ is irrational, and consequently that the length of the hypotenuse of a right triangle of side length 1, a very ordinary and real object, can not be measured as a ratio was received as a shock.

The divide between rational and irrational real numbers had raised the simple issue of just what is an irrational number, and so the quest to find mathematical entities with which every reasonable measurement is possible (even if only in theory) had begun. Somewhat astonishingly, numerous centuries have passed before an answer was given in the late 19th century. The newly found models lay to rest the need for a definition and allowed for the first time for a thorough investigation of the real number line, and, quite unexpectedly, fundamental surprises with significant ramifications were uncovered.

During the prehistoric era of the real numbers, that is those days following the Pythagoreans' discovery of the irrationality of $\sqrt{2}$ but preceding any formal construction of the reals, scientists had to cope with the reality of an ever growing list of irrational numbers (to which e was added in 1737 by Euler and π in 1761 by Lambert), the ever increasing difficulty in discerning between the rational and the irrational (it is still unknown for instance whether $e + \pi$ is rational or not), and the constant feeling of incompleteness due to the fact that no mathematical system

has been found yet that truly captures the real line. Of course, that did not stop science from progressing. After all, mathematics is merely a modeling tool for the working scientist and as long as the rough idea is good enough to work with, one does not necessarily need be discouraged by the lack of rigorous details.

Newton and Leibniz certainly were not deterred by the non-existent foundations when they developed calculus. They employed not only the real numbers but also infinitesimals - elusive numbers having the property of being positive, yet smaller than any number of the form $1/n$, for all $n \geq 1$. The scientific revolution embodied in the work of Newton and Leibniz drew immense attention from the scientific community, among which the famous quote from Berkeley's "The Analyst" regarding infinitesimals as the ghosts of departed quantities, expressing the growing discomfort at the lack of rigorous foundations. For the first time in the prehistory of the real number system significant disagreement was encountered on what did constitute a real number, and what did not. It can be said with a fair amount of certainty that the growing need to provide rigorous foundations for calculus and the increased mathematical sophistication spawned by calculus had a decisive role in the discovery of rigorous models of the real number system.

2.2. Constructions of the real numbers. Prominent figures such as Bolzano and Weierstrass attempted to construct the real numbers, with only partial success. The first correct models, and also the most commonly presented constructions in modern textbooks, were given by Dedekind (1872) and by Cantor (1873) using, respectively, cuts of rational numbers and Cauchy sequences of rational numbers. Bachmann's less well-known construction, employing nests of rational intervals, was given around the same time (1892). This trio of constructions is indicative of the fact that the major obstacle for previous generations in obtaining a rigorous definition of the real numbers was technological - the mathematical tools and techniques of calculus paved the way for three mathematicians to come up with three different solutions to the same problem.

At long last, the real numbers were born (nearly as triplets) and for a long while, about a century, no other constructions were given. Of course, there was no pressing need for more constructions, but nonetheless from 1960 till today some 16 other constructions have been given (excluding the one presented in this work). The interested reader is referred to [22] for a comprehensive survey of most, if not all, constructions of the real numbers found in the literature.

Modern textbooks often describe the real numbers axiomatically, simply by listing the axioms of a complete ordered field. It is not hard at all to prove that any two structures satisfying these axioms are isomorphic. In other words, the theory of complete ordered fields (which is a second-order theory) is categorical. The axiomatic approach is convenient enough to develop all of calculus and thus, in a sense, the sole purpose of exhibiting an actual model of the axioms is to ease one's suspicions (if any) that perhaps a contradiction is lurking underneath the surface. It is seldom the case that one uses the particularities of any given model in order to actually prove anything of interest. Once the axioms are verified, the details of the construction have served their purpose in establishing the relative consistency of the axioms, and are promptly forgotten.

The review below takes a critical look at Cantor's and Dedekind's constructions.

Cantor's construction. Cantor presented his construction of the real numbers by means of Cauchy sequences in [5]. We present here the construction only.

Consider the collection S of all rational sequences, i.e., all sequences $(a_n)_{n \geq 1}$ where $a_n \in \mathbb{Q}$. Declare such a sequence to be a *null sequence* if its limit is 0, a condition given purely in terms of rational numbers by the condition that for all rational $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for all $n \geq n_0$. The relation $(a_n) \sim (b_n)$ precisely when $(a_n - b_n)$ is a null sequence, is an equivalence relation on S , and the set of real numbers is the quotient S/\sim .

The construction quite evidently requires quotienting, though in a rather standard form. However, it must be remembered that the student encountering the reals in this fashion in a first rigorous analysis course is not accustomed to the mental juggling of equivalence classes. Consequently, even though the algebraic properties of the reals are deduced quite straightforwardly from the corresponding properties of \mathbb{Q} , pedagogically, it is questionable how effective it is for the novice to be confronted with such a complicated apparatus, whereby a real number is a set of sequences of rational numbers.

Dedekind's construction. Dedekind presented his construction of the real numbers in terms of sections, or cuts, of rational numbers in [8]. Again, we present the construction itself, mentioning that few texts actually go through the painful process of verifying all of the claims required for validating the construction. Full details, spanning numerous pages, can be found in Landau's [15], possibly the only text to actually prove all of the details.

A Dedekind cut (L, R) consists of two non-empty sets L and R which partition the rational numbers, with $x < y$ for all $x \in L$ and $y \in R$. Every rational number q determines two cuts; one where q is the largest element in L , and one where it is the smallest element in R . To avoid double representations, an arbitrary choice must be made: requiring L does not have a largest element (or, essentially equivalently, that R does have a smallest element). The set of real numbers is then the set of all Dedekind cuts.

The construction has a rather geometric flavour, addressing the incompleteness of the rationals directly. Moreover, it is an easy exercise to explicitly construct a Dedekind cut which does not correspond to any rational number (e.g., $R = \{x \in \mathbb{Q} \mid x^2 > 2\}$), and so it is immediate that one obtains new entities which were not there before (a similar demonstration using Cantor's construction is somewhat contrived, and perhaps less impressive, due to the algebraic nature of Cantor's construction versus the geometric nature of Dedekind's). However, the amount of detail required for a complete verification of the construction is staggering. In a sense, the student is required to exchange one's belief that a model of the real numbers exists by the belief that a proof that Dedekind's construction is valid exists. Nobody really expects anybody to go through the entire proof.

There really is nothing simple in the passage from \mathbb{Q} to \mathbb{R} . The discussions above should make it clear that neither Cantor's nor Dedekind's construction of the real numbers achieves the goal of introducing the real numbers rigorously and palatably at the same time. The difficulties present in one construction are somewhat complementary to those in the other construction, but each approach retains a considerable

amount of technical, conceptual, and pedagogical caveats. We conclude this discussion by presenting further criticism of Cantor’s and Dedekind’s constructions, voiced by Halmos and Conway.

In [13] Halmos reproachfully writes:

“As far as Dedekind cuts are concerned, we abstractionists have been arguing against them for a long time; it’s not quite honest to dump them in our laps and then accuse us of nurturing them. They are a historical accident. Most students of mathematics learn them as the first logically coherent way of constructing a complete ordered field, but, so far as I know, they are out of fashion by now, or in any event they ought to be. A Dedekind cut is a very narrowly focused concept. It can be generalized to certain kinds of ordered sets, but that possibility is of interest to specialists only. I am firmly convinced that one can be a broadly cultured, creative mathematician without knowing what a Dedekind cut is. Equivalence classes of Cauchy sequences are easier to understand, and, three cheers, they are more algorithmic. The important thing from the point of view of abstract mathematics, however, is that sequences are “cleaner” than cuts, more widely applicable, and more beautiful, and more structurally pertinent to the study of analysis.”

We refer to Conway’s words from [7], where the difficulties inherent to Dedekind’s construction are discussed:

“In practice the main problem is to avoid tedious case discussions. [Nobody can seriously pretend that he has ever discussed even eight cases in such a theorem – yet I have seen a presentation in which one theorem actually had 64 cases!] Now if we define \mathbb{R} in terms of Dedekind sections in \mathbb{Q} , then there are at least four cases in the definition of the product xy according to the signs of x and y [And zero often requires special treatment!]. This entails eight cases in the associative law $(xy)z = x(yz)$ and strictly more in the distributive law $(x+y)z = xz+yz$ (since we must consider the sign of $x+y$). Of course an elegant treatment will manage to discuss several cases at once, but one has to work very hard to find such a treatment.”

Shortly afterwards Conway provides the following criticism of Cantor’s construction, indicating its pedagogical difficulties:

“[The reader should be cautioned about difficulties in regarding the construction of the reals as a particular case of the completion of a metric space. If we take this line, we plainly must not start by defining a metric space as one with a real-valued metric! So initially we must allow only rational values for the metric. But then we are

faced with a problem that the metric on the completion must be allowed to have arbitrary real values!

Of course, the problem here is not actually insoluble, the answer being that the completion of a space whose metric takes values in a field \mathbb{F} is one whose metric takes values in the completion of \mathbb{F} . But there are still sufficient problems in making this approach coherent to make one feel that it is simpler to first produce \mathbb{R} from \mathbb{Q} , and later repeat the argument when one comes to complete an arbitrary metric space, and of course this destroys the economy of the approach. My own feeling is that in any case the apparatus of Cauchy sequences is logically too complicated for the simple passage from \mathbb{Q} to \mathbb{R} – one should surely wait until one has the real numbers before doing a piece of analysis!]”

For a detailed account of the historical development surrounding these constructions of the real numbers (and much more) the reader is referred to [21]. In the rest of this section we outline some aspects of the interplay between the study of the real numbers and modern mathematical developments, primarily analysis, set theory, and logic.

2.3. Transcendental numbers. Advances in the techniques of analysis led to considerable achievements on a superficially simple question regarding the nature of irrational numbers. The first confirmed examples of irrational numbers, i.e., $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. all belong to a family of irrational numbers that are quite easy to verify. Namely, if $k \geq 2$ is an integer and $n \geq 2$ is an integer not of the form m^k , $m \in \mathbb{N}$, then $\sqrt[k]{n}$ is irrational. The proof is an easy consequence of the fundamental theorem of arithmetic. However, every irrational number of the form $\sqrt[k]{n}$ satisfies an algebraic equation with integer coefficients, namely $x^k - n = 0$. Another family of real numbers easily proved to be irrational are numbers of the form $\log_a b$, for suitable values of a and b . Again, these numbers are easily shown to satisfy a polynomial relation with integer coefficients. A real number α which is the root of a polynomial with integer coefficients is called an *algebraic* number. It is immediate that any rational number is algebraic, the latter being thus a natural generalisation of the former. The above examples illustrate that numbers that are easily shown to be irrational tend to also be algebraic. The problem of existence of transcendental numbers, i.e., non-algebraic numbers, was open until 1844 when Liouville constructed the first example of a transcendental number.

Liouville’s method is analytical and in fact produces a whole family of transcendental numbers. It relies on the following lemma, which is the heart of Liouville’s construction.

Lemma (Liouville’s Lemma). *If α is an irrational algebraic number which is the*

root of a polynomial of degree $n > 0$ with integer coefficients, then a real number $L > 0$ exists such that

$$\left| \alpha - \frac{p}{q} \right| > \frac{L}{q^n}$$

holds for all integers p, q , with $q > 0$.

A proof, which is only of moderate difficulty, can be found in [17]. The theorem may be interpreted as saying that an algebraic number is either rational, or cannot be very well approximated by a rational. It is thus immediate that any irrational number α for which there exist sequences $(p_n)_{n \geq 1}$ and $(q_n)_{n \geq 1}$ of integers, with $q_n \geq 2$ for all $n \geq 1$, and such that

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^n}$$

must be transcendental. Numbers satisfying this condition are called *Liouville numbers*. The existence of Liouville numbers, and thus of transcendental numbers, is easily established, namely

$$\sum_{n=0}^{\infty} \frac{1}{10^{n!}}$$

is a Liouville number.

Liouville's result is of rather limited use when trying to settle the status of naturally occurring numbers such as e or π . The proof that e is transcendental was given in 1873 by Hermite, while the fact that π is transcendental was established in 1882 by Lindemann, both using more sophisticated techniques.

The theory of measures of irrationality and transcendental number theory carry these results much further. And yet, despite significant advances, it is still unknown whether $\pi + e$ is transcendental or not.

2.4. The uncountability of \mathbb{R} and the abundance of transcendental numbers. Numerous proofs of the uncountability of the real numbers are well-known (see, e.g., [3] for an unorthodox proof) and it is not the place here to repeat any of them (a proof utilizing the construction presented in this work is given in Theorem 5.1 below). Instead, we contemplate briefly the inherent difficulties with infinite quantities, and concentrate on the impact of Cantor's famous uncountability result.

The concept of infinity posed dramatic challenges to some of the greatest contributors to the development of science. Galileo in *Two New Sciences* presents his point of view on the matter through a discussion involving Salviati, Simplicio, and Sagredo, where Simplicio, confronted with some simple geometric observations about line segments, states that:

“Here a difficulty presents itself which seems to me insoluble. Since it is clear that we may have one line greater than another, each containing an infinite number of points, we are forced to admit that, within one and the same class, we may have something greater than infinity, because the infinity of points in the long line is greater than the infinity of points in the short line. This assigning to an infinite quantity a value greater than infinity is quite beyond my comprehension.”

Interestingly, the dialogue continues with a discussion of positive integers, with Salviati explaining to Simplicio that:

“If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square,

while no square has more than one root and no root more than one square.”

in what is so remarkably close to the notion of cardinal equality as well as to the proof that $\{n^2 \mid n \in \mathbb{N}\}$ has the same cardinality as \mathbb{N} . Salviati continues:

“But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is a root of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said that there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers...”

leading Sagredo to ask:

“What then must one conclude under those circumstances?”

to which Salviati tragically responds with:

“So far as I can see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes “equal”, “greater”, and “less”, are not applicable to infinite, but only to the finite, quantities.”

Such arguments are primarily used today by the lecturer, much to her delight, in order to torment her students into acceptance of the formal consequences resulting from the unavoidable notion that two sets between which a bijection exists are equinumerous, at the cost of rejecting one’s false beliefs about infinity, rather than the other way around. Galileo is unable to reconcile the facts and chooses to resolve the situation by throwing the baby out with the bathwater - comparability is only allowed for finite quantities.

Cantor’s creation of the haven of set theory began in 1874 with his famous publication of an article in which he demonstrates the uncountability of the reals, the countability of the algebraic numbers (which receives much emphasis in the article), and thus concluding that transcendental numbers exist without explicitly presenting any particular such number. Opinions vary regarding the precise details of events leading to, and following from, Cantor’s seminal ideas. It is well-established that Kronecker held a very narrow view on what is considered proper mathematics, quite openly rejecting Cantor’s results. In any case, even if not stated quite so bluntly, the conclusion is that not only do transcendental numbers exist (which was already shown by Liouville), but that in a precise sense the vast majority of real numbers are transcendental while a tiny proportion of real numbers are algebraic.

Cantor’s work finally made arguments about infinity possible. The student coming to terms with the counterintuitive phenomena manifesting infinite sets may take comfort in the fact that remnants of Galileo’s difficulties could still be found in Weierstrass’ assertion (see [11] for a much more thorough discussion) made in the summer of 1874, during a course he gave, to the effect that

“two ‘infinitely great magnitudes’ are not comparable and can always be regarded as equal, and that applying the notion of equality to infinite magnitudes does not lead to any result.”

An assertion of equal counter progressive power as Galileo’s conclusion that only finite quantities can be compared. Cantor’s insights freed us from the shackles of such misconceptions.

2.5. Other notions of the size of \mathbb{R} - space filling curves. Much of the counter intuitive nature of the fundamentals of cardinal comparability, such as those discussed above, or the simply established fact that \mathbb{R}^n and \mathbb{R} have the same cardinality, obviously stems from the geometric extra baggage that the observer brings with her when she perceives, e.g., \mathbb{R}^2 as a plane versus viewing \mathbb{R} as a line. As Cantor’s work was being digested by the mathematical community, some immediately sought a more careful formulation of one’s intuition that \mathbb{R} is considerably smaller than \mathbb{R}^2 by introducing topological restrictions on the size comparison. There was no disputing that $[0, 1]$ and $[0, 1] \times [0, 1]$ shared the same cardinality, but surely the line segment $[0, 1]$ can not be *continuously* mapped to the square $[0, 1] \times [0, 1]$ in such a way as to completely cover it.

Directly motivated by Cantor’s results, in 1890 Peano introduced the first example showing that even this topological intuition is faulty by constructing a surjective continuous function $[0, 1] \rightarrow [0, 1] \times [0, 1]$, a so called space-filling curve. In 1891 Hilbert constructed another such curve. Faced with these results one must admit temporary defeat in turning the ‘obvious’ fact that \mathbb{R} has dimension 1 and thus is significantly smaller than \mathbb{R}^2 whose dimension is 2, into a rigorous argument. Indeed, the topological notion of dimension is, in light of the above, not at all surprisingly, a subtle issue whose elucidation required considerable effort (see, e.g., [10, 18]). Unfortunately, further discussion here will take us too far afield from the main thread of this section.

2.6. The continuum hypothesis. Cantor’s realization that the countable cardinality of \mathbb{N} is strictly smaller than the cardinality c (the continuum) of \mathbb{R} immediately raises the question as to the existence of subsets $S \subseteq \mathbb{R}$ whose cardinality lies strictly between the countable and the continuum. The standard formulation of that question is in the form known as the continuum hypothesis, stating that any subset of \mathbb{R} is either countable (finite included) or of cardinality c . The story of the unexpected resolution of the continuum hypothesis is the subject of numerous articles and books and we shall thus be very brief.

Cantor himself was quite frustrated by his inability to resolve the situation despite many attempts to prove the continuum hypothesis (it seems Cantor was convinced of its validity). By 1900, when Hilbert addressed the mathematical community, the impact of Cantor’s set theory was firmly acknowledged and it was not at all unnatural that Hilbert listed the continuum hypothesis as the first of the 23 problems aimed at directing the efforts of mathematicians in the ensuing years.

Interestingly, it is some astounding leaps in mathematical logic, due primarily to work of Gödel leading to, and resulting from, his negative answer to the second of Hilbert’s 23 problems that prepared the ground for the final resolution of the continuum hypothesis. Hilbert’s second problem calls for a finitistic proof of

the consistency of Peano's axioms of arithmetic. The impossibility of such a program was demonstrated by Gödel in 1931 in the form of his famous incompleteness theorem.

At the time of nomination of the continuum hypothesis as the opening problem in Hilbert's list, the theory of sets was still in its infancy. In some sense, it was not even born yet; the axioms of set theory were not yet formulated, as it was only in 1908 that Zermelo proposed the first of several axiomatic systems, largely fueled by Hilbert's address. With the rapid advances in logic in the first few decades of the 20th century, Gödel was able to show in 1940 that the continuum hypothesis can not be disproved from the very well-accepted Zermelo-Fraenkel axioms of set theory (with or without the axiom of choice). It would take another 23 years until Cohen proved in 1963 that the continuum hypothesis can not be proved from the Zermelo-Fraenkel axioms either, a result of tremendous importance and impact, which led to Cohen's awarding of the Fields Medal in 1966.

The continuum hypothesis is thus forever in limbo. Without a doubt such a result was not suspected by Cantor, Hilbert, or any of their contemporaries at the time the question emerged. It is wonderfully astonishing that such a seemingly simple matter as determining the nature of the cardinalities of subsets of \mathbb{R} presented a colossal challenge, served as fuel for much of the early development of logic, and required the genius of two tremendous modern figures to resolve. There is indeed nothing simple in the passage from \mathbb{Q} to \mathbb{R} .

2.7. The non-triviality of the concept of length. Healthy geometric intuition dictates that the Riemann integral $\int_0^1 f(x)dx$ of a function which is constantly 1 except at finitely many points must be equal to 1. After all, a finite number of points is a negligible amount when computing the area determined by the graph of a function, and indeed the Riemann integral has the property that it is blind to such minute changes. With the more refined understanding of infinities, seeing that the rationals are countable while the reals are not, one also expects the integral of a function $f: [0, 1] \rightarrow \mathbb{R}$ which is constantly 1 on the irrationals and constantly 0 on the rationals to have integral equal to 1, for the exact same reason as above. However, a trivial computation shows that the Riemann integral of that function does not exist. Riemann's machinery is blind to finite changes, but it is completely obliterated by infinite changes that ought to have no effect.

Lebesgue's theory of integration resulted from the need to repair this (and other) deficiencies. The idea is beautifully simple and profound, with unexpected ramifications. The Riemann integral is obtained by introducing a partition of the X -axis, estimating the area bound under the graph (let us assume all functions are non-negative) by means of rectangles. For the Lebesgue integral one starts instead with a partition of the Y -axis, and then pulling back each segment to the X -axis by means of the inverse image under f to obtain a partition of its domain. Then each of these pre-images is used as the base of a 'rectangle' in order to estimate the area under the graph. Permitting f to so interact more dynamically in the formation of the partition of its domain, versus the more static approach of the Riemann integral where the partition is forced upon the function, suggests a process more finely tuned to the needs of the function, and thus more likely to correctly capture the behaviour of more complicated functions. The undisputed triumph of Lebesgue's

theory of integration is the result of the affirmation of this suggestion in a very broad sense.

However, before Lebesgue's integral can get off the ground, one must cope with the need to measure the 'length' of the pre-images under f . Such subsets of \mathbb{R} can be quite wild, depending on the function f , and in any case they need not look anything like an interval, or even a union of intervals. Taking a step back, one can formulate a simple question - indeed one that the ancient Greeks could have entertained - namely how does one measure the length of an arbitrary subset of \mathbb{R} , where *length* is taken in the sense of a notion that meaningfully extends the familiar length of intervals. Of course, one must state the properties one expects of such a length concept $\mu: \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty]$. The following conditions are hard to object to:

- $\mu(\emptyset) = 0$
- $\mu([0, 1]) = 1$
- $\mu(\bigcup E_i) = \sum \mu(E_i)$, for all countable collections $\{E_i\}_{i \geq 1}$ of mutually disjoint subsets of \mathbb{R}
- μ is translation invariant, meaning that $\mu(r + E) = \mu(E)$, for all $E \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, where $r + E = \{r + x \mid x \in E\}$.

With such an assignment μ (hopefully uniquely determined) Lebesgue's theory can carry through whereby all functions will be integrable. However, the relentless tendency of \mathbb{R} to harbor surprises strikes again and this marvelous dream is shattered to countably many pieces by Vitali's famous example illustrating the inconsistency of the four axioms above. The construction is far from obvious, but its details are very simple, given in the following sequence of exercises:

- Declare, for real numbers x, y , that $x \sim y$ when $x - y \in \mathbb{Q}$, and prove this is an equivalence relation.
- Use the axiom of choice to construct a set $V \subseteq [0, 1]$ consisting of precisely one representative of each equivalence class $[x]$, for each $x \in [0, 1]$.
- For each rational number $q \in [0, 1)$ let $V_q = \{a + q \mid a \in V\}$, where the computation is done modulo 1, so that V_q is again a subset of $[0, 1]$.
- Note that the family $\{V_q\}_q$ is a countable (since the rationals are countable) partition of $[0, 1]$ (by the definition of the equivalence relation), and $\mu(V_q) = \mu(V)$ (since μ is translation invariant).
- It then follows that $1 = \mu([0, 1]) = \sum_q \mu(V_q) = \sum_q \mu(V)$.
- Finally, $\mu(V) = 0$ implies $\mu([0, 1]) = 0$, while $\mu(V) > 0$ implies $\mu([0, 1]) = \infty$, leading in either case to a contradiction.

The inability to consistently measure the length of all subsets of \mathbb{R} has immediate consequences. To save Lebesgue's program, one must restrict to a σ -algebra of measurable subsets of \mathbb{R} , the so called Borel measurable sets. This σ -algebra is the smallest sigma algebra containing the intervals. This unavoidable complication is resolved quite adequately, where in fact it is seen that the crucial properties of \mathbb{R} guaranteeing everything still ticks well enough is that it is a Polish space, namely a second countable metrizable topological space whose topology can be induced by a complete metric. The Borel σ -algebra is then the σ -algebra generated by the open sets, leading to the Borel hierarchy, where the position in the hierarchy of a given Borel measurable set is determined by how many times one must perform the operations of countable unions and complementations in order to obtain the set.

The theory of Polish spaces is of fundamental importance in probability theory, mathematical statistics, and descriptive set theory.

2.8. Ghosts (of departed quantities) are real! Infinitesimal numbers, those ghosts of departed quantities, apparently truly dead and disposed off due to Cauchy’s rigorous formalization of the limit concept, still had a few tricks up their sleeves. In a fantastic turn of events, the discovery of non-standard models of the natural numbers paved the way for Robinson in the early 1960’s to use the axiom of choice and to revive the long abandoned infinitesimals, promoting their status from ghosts to flesh and blood mathematical entities. We quote from Robinson’s book ([20]):

“It is shown in this book that Leibniz’s ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics. The key to our method is provided by the detailed analysis of the relation between mathematical languages and mathematical structures which lies at the bottom of contemporary model theory.”

Robinson’s original work introducing infinitesimals in a rigorous fashion used a heavy dose of higher order logic and is quite demanding to non-logicians. Since then other approaches emerged, some of which, due to the prevalent use of ultra-products in model theory, requiring only a modest amount of preparation. The hyperreals for instance can be presented with little difficulty assuming about as much knowledge of filters as is required for the construction of the reals presented below, and thus we feel it fits nicely to present it here.

The existence of a non-principal ultrafilter on \mathbb{N} follows by a simple application of Zorn’s lemma by extending the filter of cofinite subsets of \mathbb{N} . Let us fix such a filter \mathcal{F} . Let S be the set of all sequences (a_n) or real numbers and introduce the relation $(a_n) \sim (b_n)$ precisely when $\{n \in \mathbb{N} \mid a_n = b_n\} \in \mathcal{F}$. Thus, thinking of \mathcal{F} as allowing majority sets to filter through, two sequences are considered equivalent if a majority of indices sees them as equal. It is straightforward to verify that \sim is an equivalence relation on S . The importance of \mathcal{F} being non-principal is that this equivalency does not hinge on just one fixed index (so this majority democratic system is not a dictatorship). The set of hyperreals is then the quotient $\mathbb{H} = S/\sim$, and the reals are identified therein as $[(a_n)]$ where $(a_n = a)$ is a constant sequence. One can then quite straightforwardly prove that \mathbb{H} is a field and that the transfer principle applies: every first-order sentence is true in \mathbb{R} if, and only if, it is true in \mathbb{H} . A very detailed account is given in [12]. We mention that several other approaches to infinitesimals exist, some of which were developed with the ambitious objective of replacing the standard Cauchy formalism altogether. Perhaps the most pedagogically trialled of those is [14].

The hyperreal formalism makes it deceptively simple to present a flesh and blood infinitesimal. Indeed, $h = [(1/n)_{n \geq 1}]$ is positive (since the set of indices where $1/n > 0$ is \mathbb{N} , certainly a majority set) while $h < 1/m$ for any natural number m (since the set of indices where $1/n < 1/m$ is cofinite, and thus again a majority set). However, there is a price to pay for allowing infinitesimals into the world of ordinary real numbers. Consider for instance the element $[(a_n)_{n \geq 1}] = [(1, 2, 1, 2, 1, 2, 1, 2, \dots)]$. It is clearly either equal to 1 or to 2 (since the set of indices n where the claim “ $a_n = 1$ or $a_n = 2$ ” holds is \mathbb{N}), but due to the non-constructive nature of the ultrafilter \mathcal{F} , it is impossible to determine which one it is. To date, all rigorous approaches

producing systems where infinitesimals exist alongside the real numbers appear to pay a similar price in some form or another.

2.9. The tame side of the real numbers. As awe inspiring as the complexity of \mathbb{R} is, some aspects of it must be simpler than others and it seems pertinent to identify this tamer side of the picture. A natural starting point, at least from the point-of-view of model theory, is to look at the theory of \mathbb{R} , namely all sentences (in some fixed language) which are true in \mathbb{R} . For more details, and precise definitions, the reader is referred to [16]. The choice of language may, of course, wildly change the theory, since languages with more symbols have greater expressive power. Of particular importance is whether or not quantification is restricted to only be allowed over elements (i.e., first-order logic) or if it is also allowed over sets and functions (i.e., second-order logic). The second-order theory of \mathbb{R} as a field is categorical, namely any two models (within the same ambient model of sets of course) which satisfy the same second-order sentences that \mathbb{R} does is isomorphic to \mathbb{R} , and so the theory completely captures the model.

On the other hand, the first-order theory of \mathbb{R} as a field is not categorical. A *real closed field* is a field \mathbb{F} sharing the same first-order theory with \mathbb{R} as a field, namely a first-order sentence (formulated, e.g., in the language for rings) is true in \mathbb{F} if, and only if, it is true in \mathbb{R} . The field \mathbb{H} of hyperreals is an example of a real closed field. After all, by design, it has the same first-order theory as \mathbb{R} . Obviously though, the two structures are not isomorphic, and so, indeed, the first-order theory of the field \mathbb{R} is not categorical. In 1926 Artin and Schreier proved that any ordered field admits an essentially unique order field extension which is a real closed field. In a sense then the first-order theory of the field \mathbb{R} governs all ordered fields.

Among the developments in model theory in the first half of the 20th century was the concept of quantifier elimination. A theory admits quantifier elimination if any formula is equivalent to one in which no quantifiers appear. Of course, it is remarkable when a theory admits quantifier elimination and it carries important consequences. Some form of quantifier elimination in \mathbb{R} is well-known. For instance, the fact that a quadratic with real coefficients admits real roots if, and only if, its discriminant is non-negative can be restated as follows. The first-order formula $\varphi(a, b, c) = (\exists x \quad ax^2 + bx + c = 0)$ is equivalent to the first-order formula $\psi(a, b, c) = [b^2 - 4ac \geq 0]$, a formula without quantifiers. Similarly, the first-order formula in $2n + 2$ variables stating that a polynomial of degree $2n + 1$ with real coefficients has a real root is equivalent to any formula expressing a tautology.

Each of the examples above, where a formula was replaced by an equivalent quantifier free one, involves the ordering on \mathbb{R} , and this turns out not to be coincidental. In 1951 Tarski proved that $\text{Th}(\mathbb{R})$, the first-order theory of \mathbb{R} as an *ordered field* admits quantifier elimination. To be clear then, each and every first order formula φ in a language suited to speak of ordered fields is equivalent to a quantifier free first-order formula ψ in the sense that in any model of $\text{Th}(\mathbb{R})$ the formula φ holds if, and only if, ψ holds. The proof is in fact algorithmic; it describes precisely how to construct the quantifier free formula ψ . However, of course, when applied to any of the formulas from the previous paragraph, the resulting quantifier free formula will be rather cumbersome.

As a result of quantifier elimination it follows that $\text{Th}(\mathbb{R})$ is complete, decidable, and *o*-minimal. The completeness of the theory is the statement that for every

first-order sentence φ , either it or its negation is provable, and so a situation like that presented by the continuum hypothesis is ruled out. The decidability of $\text{Th}(\mathbb{R})$ is the claim that there exists an algorithm that decides in finite time for any given first-order sentence if it is true or not (of course, there is no guarantee that such a process is effective - in fact, all of the known algorithms are of immensely high computational complexity). Finally, \mathcal{o} -minimality implies that every definable set $S \subseteq \mathbb{R}$ (i.e., one which corresponds to the set of points satisfying a first-order formula) is a finite union of intervals (points included).

2.10. Computable real numbers. Finally, we touch upon the interface between the real numbers and digital computers. Turing pioneered the study of computable numbers starting in 1936. Much more than is presented here can be found in [1]. The initial steps required subtlety, and Turing had to revise his original definition of computable numbers until he arrived at the final notion, which is essentially the following one. A real number x is *computable* if there exists a computably enumerable sequence of rationals $\{r_n\}_{n \geq 1}$ that converges to x , together with an algorithmic process that determines, for any given rational $\varepsilon > 0$, an index n for which $|r_m - x| < \varepsilon$, for all $m \geq n$. Intuitively, a real number is computable if it can be described by a finite algorithm and computed effectively to any desired accuracy.

Arguably, the computable numbers are the *real* real numbers; after all, if the digits of a number can't ever be computed, does it really exist? We shall not delve into this philosophical question here. Instead we note some facts concerning the computable real numbers. Firstly, they do form a field, obviously a countable one, and in fact a real closed field, and thus they share the same first-order theory with the reals. However, the ordering on the computable reals is not computable. In more detail, for any computable real number x , if $x \neq 0$, then it can be determined computably whether $x < 0$ or $x > 0$, as follows. For $n = 1, 2, 3, 4, \dots$ produce a rational r_n with $|x - r_n| < 2^{-n}$ (this is possible by definition of computable real). As soon as an r_n was found with either $r_n \geq 2^{-n}$ or $r_n \leq -2^{-n}$, stop the computation and announce the (obvious) answer. Now, since $x \neq 0$, this process must terminate after finitely many steps with a proof of either $x > 0$ or $x < 0$. However, if $x = 0$, then the stopping criterion is never satisfied, but there is no a-priori way, at any given step of the computation, of discerning that the stopping criterion will not be met in the next step.

More generally, but essentially equivalently, given computable real numbers x and y , if they are known to be distinct, then their ordering is computable, but without that extra information, their ordering is not computable. The branch of computable analysis studies computable numbers, computable sequences of computable numbers, computable functions, etc. where this phenomenon persists and is reflected in the difficulties of obtaining computable versions for the theorems of classical analysis. For more on this lively research area, initiated by Turing (and bifurcated to similar notions and other schools of study, for instance, by Markov) the reader is referred to [19].

To conclude this excursion of ideas we note that while for classical analysis all models of the real numbers are isomorphic, and thus it really makes no essential difference which model is used, the same is not so immediately true when computability enters the picture. Indeed, when defining the concept of a computable real numbers, one would typically first consider a classical definition of the real

numbers, and then proceed to modify the definition by requiring certain aspects in it to be computable in a suitable way. There is no a-priori reason to suspect that different classical definitions will yield the same computable notions when this process is applied. Indeed, it is an early result of Robinson that the computable versions of the real numbers obtained from the classical definitions by means of Cauchy sequences, Dedekind cuts, and nested intervals do yield the same concept.

3. Preliminary Notions

We collect here some background facts about intervals and filters which are used in the construction of the real numbers in the following section. At this point we set the convention for the rest of this work that the symbols ε , δ , η , as in $\varepsilon > 0$, stand for rational numbers.

3.1. The geometry of intervals. We list elementary and easily verified geometric properties of intervals in \mathbb{Q} (which also hold for intervals in \mathbb{R}). By an *interval* I we mean a subset of \mathbb{Q} of the form $(a, b) = \{x \in \mathbb{Q} \mid a < x < b\}$, where $a, b \in \mathbb{Q}$ with $a < b$. The *length* of $I = (a, b)$ is $b - a$. Given $p \in \mathbb{Q}$ and a rational $\varepsilon > 0$ we denote the interval $(p - \varepsilon, p + \varepsilon)$ by p_ε . We note that the addition and multiplication operations of rational numbers extend element wise to addition and multiplication operations on intervals. In more detail, given intervals I and J we define

$$I + J = \{x + y \mid x \in I, y \in J\}$$

and

$$I \cdot J = IJ = \{xy \mid x \in I, y \in J\}.$$

We are interested in the geometric effect of these operations, some of which are listed, without proof, below.

Proposition 3.1. *The following properties are easily verified:*

- (1) The sum of two intervals is again an interval. In fact, $p_\varepsilon + q_\delta = (p + q)_{\varepsilon + \delta}$, for all $p, q \in \mathbb{Q}$, $\varepsilon > 0$, and $\delta > 0$.
- (2) The product of two intervals is again an interval.
- (3) Given any $p \in \mathbb{Q}$ and $\varepsilon > 0$ one has $p_\varepsilon \subseteq p_{2\varepsilon}$ for all $y \in p_\varepsilon$.
- (4) Given an interval q_ε with $q - \varepsilon > 0$ (respectively $q + \varepsilon < 0$) the set $\frac{1}{q_\varepsilon} = \{\frac{1}{x} \mid x \in q_\varepsilon\}$ is again an interval whose length is $\frac{2\varepsilon}{q^2 - \varepsilon^2}$.

3.2. Filters. The following well-known notions and facts are stated for filters in \mathbb{Q} but hold verbatim in any metric space (where the absolute value is replaced by the distance function, and the restriction on ε being rational is replaced by it being a real number). The main aim of this subsection is to establish that any proper Cauchy filter contains a unique minimal Cauchy filter.

3.2.1. Filters and bases. A *rational filter* or more simply a *filter* (since we will only consider rational filters) is a non-empty collection \mathcal{F} of subsets of \mathbb{Q} such that

- $F_1, F_2 \in \mathcal{F}$ implies $F_1 \cap F_2 \in \mathcal{F}$
- $F_2 \supseteq F_1 \in \mathcal{F}$ implies $F_2 \in \mathcal{F}$

hold for all $F_1, F_2 \subseteq \mathbb{Q}$. The second condition implies that the only filter containing \emptyset is the filter $\mathcal{P}(\mathbb{Q})$, called the *improper filter*. It is very easy to verify that the intersection of any family of filters is again a filter.

Quite often, describing a filter is facilitated by considering only part of a filter, and then adding necessary subsets to it to form a filter. In more detail, a *filter base* is a non-empty collection \mathcal{B} of subsets of \mathbb{Q} such that for all $B_1, B_2 \in \mathcal{B}$ there exists $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$. Obviously, any filter is a filter base, but not vice versa. Given a filter base \mathcal{B} the collection $\langle \mathcal{B} \rangle = \{F \subseteq \mathbb{Q} \mid F \supseteq B, B \in \mathcal{B}\}$ is easily seen to be a filter. In fact, it is the smallest filter containing \mathcal{B} and is called the filter *generated* by the filter base \mathcal{B} .

3.2.2. Cauchy and round filters. The collection of all rational filters is large and varied. We will be interested primarily in filters that, in a sense, are *concentrated*. The precise condition is called the *Cauchy condition*, stating that for every rational $\varepsilon > 0$ there exists a rational number $q \in \mathbb{Q}$ with $q_\varepsilon \in \mathcal{F}$. Any such filter \mathcal{F} is called a Cauchy filter. It is easy to see that the Cauchy condition can equivalently be formulated as follows. For every rational $\varepsilon > 0$ there exists a rational interval $I \in \mathcal{F}$ whose length does not exceed ε . These conditions will be used interchangeably according to convenience.

The Cauchy condition on a filter can be detected on a filter base for it, as follows. Say that a filter base \mathcal{B} satisfies the Cauchy condition if for every rational $\varepsilon > 0$ there exist $q \in \mathbb{Q}$ and $B \in \mathcal{B}$ with $q_\varepsilon \supseteq B$. We then say that \mathcal{B} is a *Cauchy filter base*. It is straightforward to verify that if \mathcal{B} is a Cauchy filter base, then $\langle \mathcal{B} \rangle$ is a Cauchy filter.

Given two filters \mathcal{F} and \mathcal{G} , it is said that \mathcal{G} *refines* \mathcal{F} if $\mathcal{G} \supseteq \mathcal{F}$. It is a trivial observation that if \mathcal{G} refines \mathcal{F} and \mathcal{F} is Cauchy, then \mathcal{G} is Cauchy as well, and this shows that there is no point in asking for a unique maximal Cauchy filter containing a given Cauchy filter. However, the converse situation, i.e., asking for a minimal Cauchy filter contained in a given Cauchy filter, is very interesting. To be precise, a *minimal Cauchy filter* is a Cauchy filter \mathcal{F} such that if \mathcal{G} is any Cauchy filter satisfying $\mathcal{G} \subseteq \mathcal{F}$, then $\mathcal{G} = \mathcal{F}$. Before we can show that any Cauchy filter contains a unique minimal Cauchy filter we need to introduce the concept of a round filter.

A filter \mathcal{F} is *round* if for every $F \in \mathcal{F}$ there exists a rational number $\varepsilon > 0$ such that for all $q \in \mathbb{Q}$ if $q_\varepsilon \in \mathcal{F}$, then $q_\varepsilon \subseteq F$. Equivalently, \mathcal{F} is round if for every $F \in \mathcal{F}$ there exists a rational number $\varepsilon > 0$ such that any interval $I \in \mathcal{F}$ of length not exceeding ε satisfies $I \subseteq F$.

Roundness can also be detected on bases, as follows. Say that a filter base \mathcal{B} is a *round filter base* if for every $B \in \mathcal{B}$ there exists $\varepsilon > 0$ such that if $q_\varepsilon \supseteq B'$ for some $B' \in \mathcal{B}$, then $q_\varepsilon \subseteq B$. It is immediate that if \mathcal{B} is a round filter base, then $\langle \mathcal{B} \rangle$ is a round filter.

Example 3.2. *The improper filter $\mathcal{P}(\mathbb{Q})$ was already remarked to be Cauchy. It is not round since obviously the roundness condition for $F = \emptyset$ can not be met. Further, fix a rational number $q \in \mathbb{Q}$ and consider the collection $\{F \subseteq \mathbb{Q} \mid q \in F\}$ (which is the filter generated by the filter base $\{\{q\}\}$) and the collection $\{F \subseteq \mathbb{Q} \mid q_\varepsilon \subseteq F, \varepsilon > 0\}$ (which is the filter generated by the filter base $\{q_\varepsilon \mid \varepsilon > 0, \varepsilon \in \mathbb{Q}\}$). Both filters are Cauchy filters, but, the reader may verify, the latter is round while*

the former is not. In fact, the latter is the unique minimal Cauchy filter contained in the former.

We shall see that it is no coincidence that the conjunction of the Cauchy and roundness conditions amounts to minimal Cauchy. In fact, we can immediately obtain the following result.

Proposition 3.3. *If \mathcal{F} is Cauchy and round, then \mathcal{F} is minimal Cauchy.*

Proof. Assume that $\mathcal{G} \subseteq \mathcal{F}$ is a Cauchy filter. We need to show that $\mathcal{G} = \mathcal{F}$, thus let $F \in \mathcal{F}$ be arbitrary. Since \mathcal{F} is round there exists $\varepsilon > 0$ such that if $q_\varepsilon \in \mathcal{F}$, then $q_\varepsilon \subseteq F$. Now, since \mathcal{G} is Cauchy there exists $q \in \mathbb{Q}$ with $q_\varepsilon \in \mathcal{G}$, and thus $q_\varepsilon \in \mathcal{F}$. We conclude that $q_\varepsilon \subseteq F$ and thus, by the second condition defining a filter, that $F \in \mathcal{G}$, as required. \square

Proving the converse requires a bit more work.

3.2.3. Roundification. A proper filter which is not round can canonically be sifted to yield a round filter. The details are as follows. Given a rational $\varepsilon > 0$ and a subset $F \subseteq \mathbb{Q}$, let $F_\varepsilon = \{x \in \mathbb{Q} \mid |x - y| < \varepsilon, y \in F\}$. If \mathcal{F} is a filter, then it is a simple matter to check that $\{F_\varepsilon \mid F \in \mathcal{F}, \varepsilon > 0\}$ is a filter base. The filter generated by that filter base is denoted by \mathcal{F}_\circ and is called the *roundification* of \mathcal{F} . Notice that $F \subseteq F_\varepsilon$, and thus $\mathcal{F}_\circ \subseteq \mathcal{F}$. The following result justifies the terminology.

Proposition 3.4. *If \mathcal{F} is a proper filter, then \mathcal{F}_\circ is a round filter.*

Proof. Let $G \in \mathcal{F}_\circ$ be given, i.e., $G \supseteq F_\varepsilon$ for some $F \in \mathcal{F}$ and $\varepsilon > 0$. It now suffices to find a $\delta > 0$ such that if $q_\delta \in \mathcal{F}_\circ$, then $q_\delta \subseteq F_\varepsilon$. Consider $\delta = \frac{\varepsilon}{2}$, and suppose $q_\delta \in \mathcal{F}_\circ$, i.e., $q_\delta \supseteq F'$ for some $F' \in \mathcal{F}$. Since \mathcal{F} is proper it follows that $F \cap F' \neq \emptyset$. Using any element $y \in F \cap F'$ it is now elementary that $q_\delta \subseteq F_\varepsilon$, as required. \square

Proposition 3.5. *If \mathcal{F} is a Cauchy filter, then \mathcal{F}_\circ is Cauchy as well.*

Proof. It suffices to show that $\mathcal{B} = \{F_\varepsilon \mid F \in \mathcal{F}, \varepsilon > 0\}$ is a Cauchy filter base, and thus let $\varepsilon > 0$ be given. Since \mathcal{F} is Cauchy there exists $q \in \mathbb{Q}$ with $q_{\frac{\varepsilon}{2}} \in \mathcal{F}$. Since $q_\varepsilon = (q_{\frac{\varepsilon}{2}})_{\frac{\varepsilon}{2}} \in \mathcal{B}$ the proof is complete. \square

We can now establish the converse of Proposition 3.3.

Proposition 3.6. *If \mathcal{F} is minimal Cauchy, then \mathcal{F} is round and $\mathcal{F}_\circ = \mathcal{F}$.*

Proof. Since \mathcal{F} is minimal Cauchy \mathcal{F} is proper, and thus the filter \mathcal{F}_\circ is round and Cauchy. But $\mathcal{F}_\circ \subseteq \mathcal{F}$, and so the minimality of \mathcal{F} implies $\mathcal{F} = \mathcal{F}_\circ$, a round filter. \square

We emphasize thus that we just established that a filter is minimal Cauchy if, and only if, it is Cauchy and round.

Theorem 3.7. *If \mathcal{F} is a proper Cauchy filter, then \mathcal{F}_\circ is the unique minimal Cauchy filter contained in \mathcal{F} .*

Proof. Since \mathcal{F} is proper and Cauchy it follows that \mathcal{F}_\circ is both Cauchy and round, and thus minimal Cauchy. Suppose now that $\mathcal{G} \subseteq \mathcal{F}$ is some minimal Cauchy filter. Applying the roundification process, which clearly preserves set inclusion, yields $\mathcal{G}_\circ \subseteq \mathcal{F}_\circ$. But $\mathcal{G}_\circ = \mathcal{G}$ (since \mathcal{G} is already minimal Cauchy) and the minimality condition now implies that $\mathcal{G} = \mathcal{F}_\circ$, and thus \mathcal{F}_\circ is the only minimal Cauchy filter contained in \mathcal{F} . \square

Remark 3.8. *The results presented above are completely standard. For a categorical perspective, exhibiting the roundification process as a left adjoint, see [6].*

4. Constructing the Reals

We now present the real numbers.

4.1. The set of real numbers. Having laid down the filter theoretic preliminaries in section 3.2 we immediately proceed with the definition of the set of real numbers.

Definition 4.1. Let \mathbb{R} denote the set of all minimal rational Cauchy filters. Elements of \mathbb{R} are called *real numbers* and are typically denoted by a, b, c . In particular, each real number a is a collection of subsets of rational numbers, and we will typically refer to these sets by writing $A \in a$, while typical elements of A will be denoted by $\alpha \in A$.

We emphasize that a real number a is necessarily a proper filter, and thus $\emptyset \notin a$ and consequently $A \cap A' \neq \emptyset$ for all $A, A' \in a$. Moreover, if $A \in a$, then there exists $q \in \mathbb{Q}$ and $\varepsilon > 0$ with $q_\varepsilon \in a$ and $q_\varepsilon \subseteq A$. Indeed, as minimal Cauchy filters coincide with Cauchy and round filters, a must be round and so there exists $\varepsilon > 0$ such that $q_\varepsilon \in a$ implies $q_\varepsilon \subseteq A$, for all $q \in \mathbb{Q}$. Since a is also Cauchy, at least one $q \in \mathbb{Q}$ with $q_\varepsilon \in a$ does exist.

4.2. Order. The usual ordering of the rationals extends in two ways to the set $\mathcal{P}(\mathbb{Q})$ of all subsets of \mathbb{Q} , namely universally and existentially. Given subsets $A, B \subseteq \mathbb{Q}$ we write

$$A \leq_{\forall} B$$

if

$$\forall \alpha \in A, \beta \in B: \quad \alpha \leq \beta.$$

Similarly, we write

$$A \leq_{\exists} B$$

if

$$\exists \alpha \in A, \beta \in B: \quad \alpha \leq \beta.$$

The meaning of $A <_{\forall} B$ and $A <_{\exists} B$, as well as $A >_{\exists} B$ and $A >_{\forall} B$ etc., is defined along the same lines. Notice thus that the negation of, for instance, $A <_{\exists} B$ is $B \geq_{\forall} A$. Important to the results below is the following trivial observation, whose proof is thus omitted.

Proposition 4.2. *The relation $<_{\forall}$ is transitive on $\mathcal{P}(\mathbb{Q}) \setminus \{\emptyset\}$. In more detail, $A <_{\forall} B <_{\forall} C$ implies $A <_{\forall} C$, for all non-empty $A, B, C \subseteq \mathbb{Q}$.*

Each of the relations \leq_{\forall} and \leq_{\exists} similarly extends, both universally and existentially, to $\mathcal{P}(\mathcal{P}(\mathbb{Q}))$. To be more specific, two collections $\mathcal{G}, \mathcal{H} \subseteq \mathcal{P}(\mathbb{Q})$ satisfy

$$\mathcal{G} \leq_{\forall\exists} \mathcal{H}$$

if

$$\forall G \in \mathcal{G}, H \in \mathcal{H} : G \leq_{\exists} H.$$

Similarly,

$$\mathcal{G} \leq_{\exists\forall} \mathcal{H}$$

if

$$\exists G \in \mathcal{G}, H \in \mathcal{H} : G \leq_{\forall} H.$$

The meaning of $\mathcal{G} <_{\forall\exists} \mathcal{H}$, $\mathcal{G} \geq_{\exists\forall} \mathcal{H}$, or other derived notions, is similarly defined. It is obvious that one can further extend the ordering on \mathbb{Q} to ever more complicated nested collections of rationals, however we will only require the level two extensions given above. Note that typically these extended relations are not orderings, e.g., relations $\leq_{\exists\dots}$ starting with an existential extension are rarely transitive.

Since real numbers are collections of subsets of \mathbb{Q} we thus obtain relations on the reals which we now investigate.

For the proof of the following result, which is pivotal for the rest of the construction, recall that the intersection of filters is always a filter but that the intersection of Cauchy filters need not be Cauchy.

Lemma 4.3. *The relation $\leq_{\forall\exists}$ on \mathbb{R} is antisymmetric.*

Proof. Suppose $a \leq_{\forall\exists} b$ and $b \leq_{\forall\exists} a$, and consider the filter $a \cap b$. If it can be shown that $a \cap b$ is in fact a Cauchy filter, then as a and b are minimal Cauchy filters it will follow that $a = a \cap b = b$, and with it the result. Let then $\varepsilon > 0$ be given. As a and b are Cauchy there exist $p, q \in \mathbb{Q}$ such that $p_{\frac{\varepsilon}{2}} \in a$ and $q_{\frac{\varepsilon}{2}} \in b$. We may assume, without loss of generality, that $p < q$. Since $b \leq_{\forall\exists} a$ it follows that $q_{\frac{\varepsilon}{2}} \leq_{\exists} p_{\frac{\varepsilon}{2}}$ and thus (remember that $p < q$) that $p_{\frac{\varepsilon}{2}} \cap q_{\frac{\varepsilon}{2}} \neq \emptyset$, so that we may find $r \in p_{\frac{\varepsilon}{2}} \cap q_{\frac{\varepsilon}{2}}$. It is now elementary to verify that $r_{\varepsilon} \supseteq p_{\frac{\varepsilon}{2}}$, and thus $r_{\varepsilon} \in a$. Similarly, $r_{\varepsilon} \in b$ and we may conclude, as was intended, that $a \cap b$ is Cauchy. \square

The following corollary is useful.

Theorem 4.4 (Equality Criterion For Real Numbers). *Two real numbers a and b are equal if, and only if, $A \cap B \neq \emptyset$ for all $A \in a$ and $B \in b$.*

Proof. If $a = b$, then $A \cap B \neq \emptyset$ for all $A, B \in a$ simply because a is a proper filter. Conversely, if $A \cap B \neq \emptyset$ for all $A \in a$ and $B \in b$, then any $x \in A \cap B$ demonstrates that $A \leq_{\exists} B$ and $B \leq_{\exists} A$. Consequently, $a \leq_{\forall\exists} b$ and $b \leq_{\forall\exists} a$, yielding $a = b$. \square

The stage is now set for introducing the total ordering on the reals.

Definition 4.5. For real numbers $a, b \in \mathbb{R}$ we write $x < y$ if $x <_{\exists\forall} y$.

Theorem 4.6. *$(\mathbb{R}, <)$ is a total ordering.*

Proof. To show irreflexivity, assume that $a < a$. Then $A <_{\forall} A'$ for some $A, A' \in a$, but $A \cap A' \neq \emptyset$, clearly a contradiction. Next, to show transitivity, assume $a < b < c$. Then $A <_{\forall} B'$ and $B'' <_{\forall} C$ for some $A \in a$, $B', B'' \in b$, and $C \in c$, which are all necessarily non-empty. Taking $B = B' \cap B''$, which is again non-empty, it follows that $A <_{\forall} B <_{\forall} C$ and so the transitivity of $<_{\forall}$ on $\mathcal{P}(\mathbb{Q}) \setminus \{\emptyset\}$ implies that $A <_{\forall} C$,

and so $a < c$. To show asymmetry, suppose that $a < b$ and $b < a$ both hold. Then $A <_{\forall} B$ and $B' <_{\forall} A'$ for some $A, A' \in a$ and $B, B' \in b$. Hence $A <_{\forall} B \cap B' <_{\forall} A'$ and, again, none of these sets is empty so we may conclude that $A <_{\forall} A'$. But that implies that $a < a$, which was already seen to be impossible.

The proof up to now only used the fact that the reals are modeled by filters. To complete the proof we need to show that if $a \neq b$, then either $a < b$ or $b < a$. It is here that the minimal Cauchy condition plays a role (via Lemma 4.3). Indeed, if $a \not< b$ and $b \not< a$, then $a \geq_{\forall\exists} b$ and $b \geq_{\forall\exists} a$, and thus $a = b$. \square

4.3. The embedding of \mathbb{Q} in \mathbb{R} . Every $q \in \mathbb{Q}$ gives rise to two filters. One is the *maximal principal filter* $\langle q \rangle = \{S \subseteq \mathbb{Q} \mid q \in S\}$, and the other is the *minimal principal filter* $\iota(q) = \{S \subseteq \mathbb{Q} \mid q_{\varepsilon} \subseteq S, \varepsilon > 0\}$. Clearly $\iota(q) \subseteq \langle q \rangle$ and each filter is Cauchy.

Lemma 4.7. *For all rational numbers q the filter $\iota(q)$ is a real number.*

Proof. One way to proceed is to show that $\iota(q) = \langle q \rangle_{\circ}$, the roundification of the maximal filter $\langle q \rangle$. Alternatively, we will show directly that $\iota(q)$ is a minimal Cauchy filter. Suppose that $\mathcal{F} \subseteq \iota(q)$ is a Cauchy filter and let $S \in \mathcal{F}$, namely $q_{\varepsilon} \subseteq S$ for some $\varepsilon > 0$. Our goal is to show that $S \in \mathcal{F}$. As \mathcal{F} is Cauchy, there exists $p \in \mathbb{Q}$ such that $p_{\frac{\varepsilon}{2}} \in \mathcal{F}$, and, since $\mathcal{F} \subseteq \iota(q)$, $q \in p_{\frac{\varepsilon}{2}}$. It now follows that $p_{\frac{\varepsilon}{2}} \subseteq q_{\varepsilon} \subseteq S$, and, since \mathcal{F} is a filter, $S \in \mathcal{F}$, as required. \square

We thus obtain a function $\iota : \mathbb{Q} \rightarrow \mathbb{R}$.

Proposition 4.8. *The function $\iota : \mathbb{Q} \rightarrow \mathbb{R}$ is an order embedding.*

Proof. Assume that $p < q$ are given rational numbers and let $\varepsilon = \frac{q-p}{2}$. Clearly $p_{\varepsilon} <_{\forall} q_{\varepsilon}$ and since $p_{\varepsilon} \in \iota(p)$ and $q_{\varepsilon} \in \iota(q)$, we conclude that $\iota(p) <_{\forall\exists} \iota(q)$, namely that $\iota(p) < \iota(q)$. \square

Lemma 4.9. *For all $q \in \mathbb{Q}$ and $a \in \mathbb{R}$*

- (1) $a < \iota(q)$ if, and only if, $a <_{\forall\exists} \{\{q\}\}$.
- (2) $\iota(q) < a$ if, and only if, $\{\{q\}\} <_{\forall\exists} a$.

Proof. In the \implies direction, both arguments follow the exact same pattern, i.e., if $a < \iota(q)$, then $A <_{\forall} S$ for some $A \in a$ and $S \in \iota(q)$, but then since $q \in S$ it follows that $A <_{\forall} \{q\}$, and thus $a <_{\forall\exists} \{\{q\}\}$. The arguments in the other direction are also similar to each other. Suppose $a <_{\forall\exists} \{\{q\}\}$ holds but $a < \iota(q)$ does not, i.e., either $a = \iota(q)$ or $a > \iota(q)$. From $a <_{\forall\exists} \{\{q\}\}$ it follows that there exists $A \in a$ with $A <_{\forall} \{q\}$. Suppose that $a > \iota(q)$ holds, i.e., $a >_{\forall\exists} \{\{q\}\}$. There exist then $A' \in a$ such that $\{q\} <_{\forall} A'$. Considering $A \cap A'$, which is non-empty, we have $\{q\} <_{\forall} A \cap A' <_{\forall} \{q\}$, which is nonsense. Assume now that $a = \iota(q)$. But then $q \in A$ and thus $A <_{\forall} \{q\}$ is impossible. We conclude that $a < \iota(q)$. \square

This result shows that we can quite safely abuse notation and identify $\iota(q)$ with q . However, we resist the temptation of making this identification quite yet and, for the sake of clarity, we opt for the following definition.

Definition 4.10. A real number a is a *rational real number* if there exists $q \in \mathbb{Q}$ with $a = \iota(q)$. A real number that is not a rational real number is called an *irrational real number*.

We now give an internal characterization of the rational real numbers. For a filter \mathcal{F} , the intersection $C(\mathcal{F}) = \bigcap_{F \in \mathcal{F}} F$ of all of its members is called the *core* of \mathcal{F} . The filter \mathcal{F} is said to be *free* if its core is empty.

Lemma 4.11. *Let a be a real number.*

- *a is a rational real number if, and only if, its core is a singleton set, and in that case $a = \iota(q)$ if, and only if, $C(a) = \{q\}$.*
- *a is an irrational real number if, and only if, it is a free filter.*

Proof. Firstly, we establish that the core of any Cauchy filter \mathcal{F} is either empty or a singleton set. Indeed, suppose that $p, q \in C(\mathcal{F})$ and $p < q$, and let $\varepsilon = \frac{q-p}{2}$. Since \mathcal{F} is Cauchy, there exists an $x \in \mathbb{Q}$ with $x_\varepsilon \in \mathcal{F}$. Since p and q are in the core of \mathcal{F} it follows that $p, q \in x_\varepsilon$, which by the choice of ε is impossible.

Assume now that a is rational, i.e., $a = \iota(q)$ for some $q \in \mathbb{Q}$. It is obvious from the definition of $\iota(q)$ that $q \in C(\iota(q))$, and thus necessarily $C(\iota(q)) = \{q\}$. Conversely, if $C(a) = \{q\}$ for some $q \in \mathbb{Q}$, then $q \in A \cap B$ for all $A \in a$ and $B \in \iota(q)$. It follows from Theorem 4.4 that $a = \iota(q)$. The characterization of irrational real numbers follows by contrapositives. \square

We conclude the treatment of the order on \mathbb{R} and the place of the rational real numbers within the reals by establishing density, the archimedean property, and some useful technical results.

Proposition 4.12. *The equivalence*

$$q_\varepsilon \in a \iff \iota(q - \varepsilon) < a < \iota(q + \varepsilon)$$

holds for all real numbers a and rational numbers q and $\varepsilon > 0$.

Proof. If $q_\varepsilon \in a$ then it follows from Lemma 4.11 and the obvious inequalities $\{q - \varepsilon\} <_\vee q_\varepsilon <_\vee \{q + \varepsilon\}$ that $\iota(q - \varepsilon) < a < \iota(q + \varepsilon)$. Conversely, suppose that $\iota(q - \varepsilon) < a < \iota(q + \varepsilon)$. Then, again by Lemma 4.11, there exist $A, A' \in a$ with $\{q - \varepsilon\} <_\vee A$ and $A' <_\vee \{q + \varepsilon\}$. Let $A_0 = A \cap A'$ and then $\{q - \varepsilon\} <_\vee A_0 <_\vee \{q + \varepsilon\}$, implying that $A_0 \subseteq q_\varepsilon$. Since $A_0 \in a$ it follows that $q_\varepsilon \in a$. \square

Remark 4.13. *Notice the immediate equivalent formulation of this result, namely that $q_\varepsilon \notin a$ if, and only if, either $a \leq \iota(q - \varepsilon)$ or $a \geq \iota(q + \varepsilon)$.*

Corollary 4.14 (Rational Approximations). *For all $a \in \mathbb{R}$ and $\varepsilon > 0$ there exist a rational number q and $\varepsilon > 0$ such that $\iota(q - \varepsilon) < a < \iota(q + \varepsilon)$.*

Proof. Every real number $a \in \mathbb{R}$ is a Cauchy filter, and thus for any $\varepsilon > 0$ there exists $q \in \mathbb{Q}$ with $q_\varepsilon \in a$. \square

Corollary 4.15. *The rational real numbers are dense in \mathbb{R} .*

Proof. Let $a < b$ be real numbers. There exist then $A \in a$ and $B \in b$ with $A <_\vee B$. Inside A we may find an interval $p_\varepsilon \subseteq A$ with $p_\varepsilon \in a$. Similarly, there is an interval $q_\delta \subseteq B$ with $q_\delta \in b$. Clearly, $p_\varepsilon <_\vee q_\delta$, which implies that $p + \varepsilon \leq q - \delta$. We now have that $a < \iota(p + \varepsilon) \leq \iota(q - \delta) < b$, and so at least one rational real number between a and b is found. \square

Corollary 4.16. *\mathbb{R} is an archimedean order in the sense that for all $a \in \mathbb{R}$ there exists $n \in \mathbb{N}$ with $a < \iota(n)$.*

Proof. If $a < \iota(q + \varepsilon)$, then any $n > q + \varepsilon$ will do. \square

Summarizing the results so far we see that the function $\iota : \mathbb{Q} \rightarrow \mathbb{R}$ is a dense order embedding. At this point it is a simple matter to deduce the following useful result.

Proposition 4.17. *For any real number a the following hold*

- There exists a rational number $M > 0$ and $A \in a$ with $\{-M\} <_{\vee} A <_{\vee} \{M\}$.
- $a > 0$ if, and only if, for every $A \in a$ there exists $A_+ \subseteq A$ with $A_+ >_{\vee} \{0\}$ and $A_+ \in a$.
- $a < 0$ if, and only if, for every $A \in a$ there exists $A_- \subseteq A$ with $A_- <_{\vee} \{0\}$ and $A_- \in a$.
- $a = 0$ if, and only if, $0 \in A$ for all $A \in a$.

Proof. For the first assertion, there exists $p \in \mathbb{Q}$ with $p_1 \in a$, so a choice for M is clear. If $a > 0$, then $a >_{\exists\vee} \{\{0\}\}$, so there is some $A^+ \in a$ with $A^+ >_{\vee} \{0\}$. Given any $A \in a$ one may then take $A_+ = A \cap A^+$, which fulfills the required condition. Conversely, the condition on A_+ immediately implies that $a >_{\exists\vee} \{\{0\}\}$, and hence $a > 0$. The proof for negative numbers is similar, and the characterization of $a = 0$ is just a restatement of the characterization of all rational real numbers in terms of cores. \square

The following technically sharper result on positive and negative real numbers is also useful.

Proposition 4.18. *If $a > 0$ is a real number, then there exists a rational number $\eta > 0$ and a rational number $\delta > 0$ such that $p_{\delta'} \in a$ implies $p_{\delta'} >_{\vee} \{\eta\}$, for all $p \in \mathbb{Q}$ and $0 < \delta' \leq \delta$. A similar result holds for negative real numbers.*

Proof. Let $\eta > 0$ be a rational number with $a > \iota(3\eta)$. Then there exists $A \in a$ such that $A >_{\vee} \{3\eta\}$. Let $\delta = \eta$. Then, if $p_{\delta'} \in a$ with $0 < \delta' \leq \delta$, then $p_{\delta'} \cap A \neq \emptyset$, implying $x > 3\eta$ for some $x \in p_{\delta'}$, and as the length of $p_{\delta'}$ is $2\delta' < 2\eta$ it follows that $p_{\delta'} >_{\vee} \{\eta\}$, as required. \square

4.4. Arithmetic. The addition operation of the rational numbers extends element-wise to sets $A, B \subseteq \mathbb{Q}$, i.e., we define $A + B = \{\alpha + \beta \mid \alpha \in A, \beta \in B\}$. Further, for arbitrary collections \mathcal{F} and \mathcal{G} of subsets of rational numbers, let $\mathcal{F} \oplus \mathcal{G} = \{A + B \mid A \in \mathcal{F}, B \in \mathcal{G}\}$. In particular, for real numbers $a, b \in \mathbb{R}$, we have the collection $a \oplus b = \{A + B \mid A \in a, B \in b\}$. Similarly, multiplication is also extended via $A \cdot B = AB = \{\alpha\beta \mid \alpha \in A, \beta \in B\}$ and $\mathcal{F} \odot \mathcal{G} = \{AB \mid A \in \mathcal{F}, B \in \mathcal{G}\}$, and in particular for real numbers then $a \odot b = \{AB \mid A \in a, B \in b\}$. For $p \in \mathbb{Q}$ and $B \subseteq \mathbb{Q}$ we write $p + B$ as shorthand for $\{p\} + B$, and similarly $p \cdot B$ for $\{p\} \cdot B$.

Proposition 4.19. *For all real numbers $a, b \in \mathbb{R}$, the collections $a \oplus b$ and $a \odot b$ are Cauchy filter bases.*

Proof. Each collection is clearly non-empty. The condition for filter base is verified by noting that for all $A, A' \in a$ and $B, B' \in b$

$$\begin{aligned} (A \cap A') + (B \cap B') &\subseteq (A + B) \cap (A' + B') \\ (A \cap A')(B \cap B') &\subseteq (AB) \cap (A'B') \end{aligned}$$

together with the fact that $A \cap A' \in a$ and $B \cap B' \in b$. The Cauchy condition for $a \oplus b$ is immediate; for $\varepsilon > 0$ there exist $p, q \in \mathbb{Q}$ with $p_{\frac{\varepsilon}{2}} \in a$ and $q_{\frac{\varepsilon}{2}} \in b$, and

then $(p + q)_\varepsilon = p_{\frac{\varepsilon}{2}} + q_{\frac{\varepsilon}{2}} \in a \oplus b$. As for $a \odot b$, fix $\varepsilon > 0$, and a natural number M together with $A \in a$ and $B \in b$ with $\{-M\} \leq_{\forall} A, B \leq_{\forall} \{M\}$, and let $\delta = \frac{\varepsilon}{M+2\varepsilon}$. As a and b are Cauchy, there exist $p, q \in \mathbb{Q}$ with $p_\delta \in a$ and $q_\delta \in b$. In particular, both intervals are in the range $(-M - 2\varepsilon, M + 2\varepsilon)$ (since p_a intersects A , and p_b intersects B) and thus the interval $p_\delta q_\delta$ has length bounded by $(M + 2\varepsilon) \frac{\varepsilon}{(M+2\varepsilon)} = \varepsilon$, as required. \square

Definition 4.20. Given real numbers $a, b \in \mathbb{R}$, their *sum* is $a + b = \langle a \oplus b \rangle$ and their *product* is $ab = \langle a \odot b \rangle$. In more detail, a subset $C \subseteq \mathbb{Q}$ satisfies $C \in a + b$ (respectively $C \in ab$) precisely when there exist $A \in a$ and $B \in b$ with $C \supseteq A + B$ (respectively $C \supseteq AB$).

We will denote $0 = \iota(0)$, unless doing so may cause confusion.

Proposition 4.21. *The equalities $a \cdot \iota(0) = \iota(0) = \iota(0) \cdot a$ hold for all $a \in \mathbb{R}$.*

Proof. Suppose that $B \in a \cdot \iota(0)$. Then $B \supseteq A \cdot 0_\varepsilon$ for some $A \in a$ and $\varepsilon > 0$. Further, $A \neq \emptyset$ and for any $\alpha \in A$ we have $B \supseteq \alpha \cdot 0_\varepsilon = 0_{\alpha\varepsilon}$, and thus $B \in \iota(0)$, showing that $a \cdot \iota(0) \subseteq \iota(0)$. In the other direction, if $B \in \iota(0)$, then $B \supseteq 0_\varepsilon$, with $\varepsilon > 0$. Let now $A \in a$ and $M > 0$ with $\{-M\} <_{\forall} A <_{\forall} \{M\}$, and consider $\delta = \frac{\varepsilon}{M}$. It follows easily that $0_\varepsilon \supseteq A \cdot 0_\delta$, and thus $0_\varepsilon \in a \cdot \iota(0)$, as needed. The proof that $\iota(0) = \iota(0) \cdot a$ is similar. \square

Lemma 4.22. *The sum and product of any two real numbers a and b are real numbers.*

Proof. As $a \oplus b$ and $a \odot b$ are Cauchy filter bases, it follows that $a + b$ and ab are Cauchy filters so it only remains to be shown that each of these filters is also round. We start with $a + b$. Let $C \in a + b$, i.e., $C \supseteq A + B$ for some $A \in a$ and $B \in b$. Our goal is to find $\delta > 0$ such that $p_\delta \in a + b$ implies $p_\delta \subseteq C$, for all $p \in \mathbb{Q}$. Since a is round there exists $\varepsilon' > 0$ for which $x_{\varepsilon'} \in a$ implies $x_{\varepsilon'} \subseteq A$, for all $x \in \mathbb{Q}$. Similarly, there exists $\varepsilon'' > 0$ such that $x_{\varepsilon''} \in b$ implies $x_{\varepsilon''} \subseteq B$, for all $x \in \mathbb{Q}$. Let $\delta = \min\{\varepsilon', \varepsilon''\}$. Suppose now that $p_\delta \in a + b$ holds for some $p \in \mathbb{Q}$, namely $p_\delta \supseteq A' + B'$ for some $A' \in a$ and $B' \in b$. We will conclude the proof by showing that $p_\delta \subseteq C$, which will be achieved by showing that $p_\delta \subseteq A + B$. It is easily seen that $p_\delta \subseteq y_{2\delta}$ for any $y \in p_\delta$, and so it suffices to find $y \in A' + B'$ with $y_{2\delta} \subseteq A + B$. For all $\alpha' \in A'$ and $\beta' \in B'$ we have that $(\alpha' + \beta')_{2\delta} = \alpha'_\delta + \beta'_\delta \subseteq \alpha'_{\varepsilon_1} + \beta'_{\varepsilon_2}$, so the problem is now reduced to finding $\alpha' \in A'$ and $\beta' \in B'$ with $\alpha'_{\varepsilon_1} \subseteq A$ and $\beta'_{\varepsilon_2} \subseteq B$. The existence of such elements is verified as follows. Let $\alpha' \in \mathbb{Q}$ and $r > 0$ with $\alpha'_r \in a$ and $\alpha'_r \subseteq A'$. Since $A' + B' \subseteq p_\delta$ it is seen that α'_r translates (by any element in B') into p_δ , and thus $r \leq \delta$. We now know that $\alpha'_r \subseteq \alpha'_\delta \subseteq \alpha'_{\varepsilon'}$, and thus that $\alpha'_{\varepsilon'} \in a$, and from the choice of ε' we conclude that $\alpha'_{\varepsilon'} \subseteq A$. The existence of $\beta' \in B'$ with $\beta'_{\varepsilon''} \subseteq B$ follows similarly, and the proof is now complete. The proof for the product follows along the same lines, utilizing the fact that for every real number c there exists a natural number M and $C \in c$ with $\{-M\} <_{\forall} C <_{\forall} \{M\}$ in order to obtain correct bounds. We omit the details. \square

Proposition 4.23. *If $a = \iota(p)$ is a rational real number and b is any real number, then $\iota(p) + b = \{p + B \mid B \in b\}$ and $\iota(p) \cdot b = \{p \cdot B \mid B \in b\}$.*

Proof. The fact that the collection $\{p + B \mid B \in b\}$ is a real number, i.e., that it is a filter, that it is Cauchy, and that it is round, are immediate. To show that $\{p + B \mid B \in b\} = \iota(p) + b$ we apply Theorem 4.4. Given an arbitrary $p + B$ and an arbitrary $x \in \iota(p) + b$, namely $x \supseteq A + B'$ with $A \in \iota(p)$ and $B' \in b$, we need to show that $(p + B) \cap (A + B') \neq \emptyset$. But since $p \in A$ for all $A \in \iota(p)$ and since $B \cap B' \neq \emptyset$ that claim is obvious. For the multiplicative part of the claim, note that the case $p = 0$ is Proposition 4.21. For $p \neq 0$ it is straightforward that $\{p \cdot B \mid B \in b\}$ is a real number and the rest of the proof is virtually the same as the additive claim. \square

Theorem 4.24. *With addition and multiplication of real numbers, \mathbb{R} is a commutative ring with unity.*

Proof. Let $a, b, c \in \mathbb{R}$ be given. We note first that

- $\langle a \oplus (b + c) \rangle = \langle a \oplus (b \oplus c) \rangle$
- $\langle (a + b) \oplus c \rangle = \langle (a \oplus b) \oplus c \rangle$
- $\langle a \odot (bc) \rangle = \langle a \odot (b \odot c) \rangle$
- $\langle (ab) \odot c \rangle = \langle (a \odot b) \odot c \rangle$
- $\langle a \odot (b + c) \rangle = \langle a \odot (b \oplus c) \rangle$.

Indeed, all of these equalities are established by essentially the same argument, so we only verify the first one. Since $b \oplus c \subseteq b + c$, one of the inclusions is trivial. For the other inclusion, a typical element X in $\langle a \oplus (b + c) \rangle$ is a subset of \mathbb{Q} with $X \supseteq A + Y$ for some $A \in a$ and $Y \in b + c$. But then Y itself contains a set of the form $B + C$ for $B \in b$ and $C \in c$, and thus $X \supseteq A + (B + C)$, and is thus in $a \oplus (b \oplus c)$. Associativity of $+$ now follows at once since

$$a + (b + c) = \langle a \oplus (b + c) \rangle = \langle a \oplus (b \oplus c) \rangle = \langle (a \oplus b) \oplus c \rangle = \langle (a + b) \oplus c \rangle = (a + b) + c$$

as $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ is immediate since obviously $(A + B) + C = A + (B + C)$ holds for all subsets of \mathbb{Q} . Repetition of this argument shows that multiplication is associative, that both addition and multiplication are commutative, and the distributivity law. Proposition 4.23 implies at once the neutrality of $0 = \iota(0)$ and that $1 = \iota(1)$ is a multiplicative identity element. Finally, for the existence of additive inverses, consider a real number a and let $b = \{-A \mid A \in a\}$, where $-A = \{-\alpha \mid \alpha \in A\}$. Obviously, b is a real number and we now show that $a + b = 0$. By Lemma 4.11 it suffices to show that $0 \in C$ for all $C \in a + b$. Indeed, for such a C there exists $A, A' \in a$ with $C \supseteq A + (-A') \supseteq (A \cap A') + (-(A \cap A'))$, a set which certainly contains 0 since $A \cap A' \neq \emptyset$. \square

Theorem 4.25. \mathbb{R} *With addition and multiplication is a field.*

Proof. Fix a real number $a > 0$, for which we shall present an inverse a^{-1} . For an arbitrary $A \subseteq \mathbb{Q}$ let $\frac{1}{A} = \{\frac{1}{\alpha} \mid \alpha \in A, \alpha \neq 0\}$. Noting that $\frac{1}{A} \cap \frac{1}{A'} = \frac{1}{A \cap A'}$, it follows at once that $\mathcal{B} = \{\frac{1}{A} \mid A \in a\}$ is a filter base. We proceed to show that it is Cauchy, so let us fix an $\varepsilon > 0$. By Proposition 4.18 there exists $\eta > 0$ and $\delta > 0$ such that $p_{\delta'} \in a$ implies $p_{\delta'} \supset \{\eta\}$, for all $p \in \mathbb{Q}$ and $0 < \delta' \leq \delta$. It then follows that $\frac{1}{p_{\delta'}}$ is an interval whose length is $\frac{2\delta'}{(p-\delta')(p+\delta')} \leq \frac{2\delta'}{\eta^2} < \varepsilon$, for a sufficiently small $\delta' > 0$. The existence of some $p_{\delta'} \in a$ is guaranteed since a is Cauchy.

We may now define $a^{-1} = \langle \frac{1}{A} \mid A \in a \rangle$, the generated filter (with slight abuse of notation), which is thus Cauchy. It is a bit tedious to show directly that \mathcal{B} is also

a round filter base. To avoid these details, and since we are only interested in the existence of a multiplicative inverse, let us consider $a^{-1} = \langle \mathcal{B} \rangle_{\circ}$, the roundification of the generated filter, which is thus both Cauchy and round, and hence a real number. To show that $a \cdot a^{-1} = 1 = \iota(1)$ it suffices to compute the core and appeal to Lemma 4.11. Indeed, since $\langle \mathcal{B} \rangle_{\circ} \subseteq \langle \mathcal{B} \rangle$, given $C \in a \cdot a^{-1}$ there exist $A, A' \in a$ such that $C \supseteq A \cdot \frac{1}{A'}$ which contains $(A \cap A') \cdot \frac{1}{A \cap A'}$, which itself clearly contains 1 since $A \cap A' \neq \emptyset$, and in fact contains at least two elements, one of which is not 0.

The proof for $a < 0$ is similar. \square

Theorem 4.26. \mathbb{R} with addition and multiplication is an ordered field.

Proof. Let $a, b, c \in \mathbb{R}$ with $a \leq b$. We have to show that $a + c \leq b + c$ and, if $c > 0$, that $ac \leq bc$. Indeed, given arbitrary $D \in a + c$ and $D' \in b + c$ there exist $A \in a$, $B \in b$, and $C, C' \in c$ with $D \supseteq A + C$ and $D' \supseteq B + C'$. Further, since $a \leq_{\forall\exists} b$, it holds that $A \leq_{\exists} B$, so that $\alpha \leq \beta$ for some $\alpha \in A$ and $\beta \in B$. Since $C \cap C' \neq \emptyset$ let $\gamma \in C \cap C'$. Then $\alpha + \gamma \leq \beta + \gamma$, showing that $A + C \leq_{\exists} B + C'$, and thus that $D \leq_{\exists} D'$. As D and D' were arbitrary we showed that $a + c \leq_{\forall\exists} b + c$, as required.

The argument for showing that $ac \leq bc$ under the further condition $c > 0$ is similar. Firstly, since $c > 0$ there exists $C_+ \in c$ with $C_+ >_{\forall} \{0\}$. Now, given arbitrary $D \in ac$ and $D' \in bc$ there exist $A \in a$, $B \in b$, and $C, C' \in c$ with $D \supseteq AC$ and $D' \supseteq BC'$. As above, we have $\alpha \leq \beta$ for some $\alpha \in A$ and $\beta \in B$. As $C \cap C' \cap C_+$ is non-empty, let $\gamma \in C \cap C' \cap C_+$. Then $\alpha\gamma \leq \beta\gamma$, showing that $AC \leq_{\exists} BC'$, and thus that $D \leq_{\exists} D'$, which were arbitrary and thus $ac \leq_{\forall\exists} bc$. The proof is complete. \square

Corollary 4.27. The canonical embedding $\iota : \mathbb{Q} \rightarrow \mathbb{R}$ is a field homomorphism.

Proof. We have to show that $\iota(p + q) = \iota(p) + \iota(q)$ and that $\iota(pq) = \iota(p)\iota(q)$, for all $p, q \in \mathbb{Q}$. Indeed, since $C(\iota(p)) = \{p\}$ and $C(\iota(q)) = \{q\}$ it follows at once that $p + q \in C(\iota(p) + \iota(q))$ and that $pq \in C(\iota(p)\iota(q))$. The claim now follows by Lemma 4.11. \square

4.5. Completeness. We now establish the completeness property of the reals. Let us fix a non-empty set \mathcal{A} of real numbers and assume that it is bounded above by some real number c . Consider the collection $\mathbb{A} = \{p_{\varepsilon} \mid p_{\varepsilon} \in a_0, a_0 \in \mathcal{A}\}$, which represents an attempt to collate all of \mathcal{A} into a single real number. However, this collection fails to be a filter. We thus refine it by considering the collection $\mathcal{B} = \{p_{\varepsilon} \in \mathbb{A} \mid a < \iota(p + \varepsilon), \forall a \in \mathcal{A}\}$, which is non-empty since \mathcal{A} is non-empty and bounded above. Intuitively, the condition sifts away those elements in \mathbb{A} which lie too far below in \mathcal{A} . The proof of completeness proceeds in two steps:

- (1) Establish that \mathcal{B} is a Cauchy filter base.
- (2) Prove that $b = \langle \mathcal{B} \rangle_{\circ}$ is the least upper bound of \mathcal{A} .

Let us first tend to the second task as it is quite straightforward. Working under the supposition that \mathcal{B} is a Cauchy filter base, it follows that $\langle \mathcal{B} \rangle$, the generated filter, is a Cauchy filter. The filter $b = \langle \mathcal{B} \rangle_{\circ}$ is thus Cauchy and round, namely a real number. Recalling that the roundification of a filter \mathcal{F} always yields a subfilter of \mathcal{F} we have that $b \subseteq \langle \mathcal{B} \rangle$. Consequently, for every $B \in b$ there exists $q \in \mathbb{Q}$ and $\varepsilon > 0$ with $B \supseteq q_{\varepsilon} \in \mathcal{B}$. We call any such q_{ε} a *witnessing interval* for B .

Given an arbitrary $a_0 \in \mathcal{A}$ we need to show that $a_0 \leq_{\forall\exists} b$, so let us fix $A \in a_0$ and $B \in b$, and we need to establish that $A \leq_{\exists} B$. Let q_{ε} be a witnessing interval

for B . Since $q_\varepsilon \in \mathcal{B}$ it follows that $a_0 < \iota(q + \varepsilon)$. It now follows that there exists $A' \in a_0$ with $A' <_{\forall} \{q + \varepsilon\}$, and let $A_0 = A \cap A'$, which is in a_0 and thus non-empty. It suffices to show now that $A_0 \leq_{\exists} q_\varepsilon$. But for any $\alpha \in A_0$ it follows that $\alpha < q + \varepsilon$, and thus $\alpha \leq y$ for some $y \in q_\varepsilon$, as required. To complete the argument that b is the least upper bound, suppose that c is any upper bound of \mathcal{A} and assume that $c < b$. There exist then $C \in c$ and $B \in b$ with $C <_{\forall} B$. Taking a witnessing interval q_ε for B we have that $C <_{\forall} q_\varepsilon$, and $q_\varepsilon \in a_0$ for some $a_0 \in \mathcal{A}$. However, c is an upper bound of \mathcal{A} and thus $c \geq a_0$, namely $c \geq_{\forall \exists} a_0$. It follows that $C \geq_{\exists} q_\varepsilon$, clearly contradicting $C <_{\forall} q_\varepsilon$.

It now remains to show that \mathcal{B} is a Cauchy filter base. Firstly, \mathcal{B} is a filter base as follows. Suppose $p_\varepsilon, q_\delta \in \mathcal{B}$, with $p_\varepsilon \in a'$ and $q_\delta \in a''$, and, without loss of generality, $a' \leq a''$. Now, the intersection $p_\varepsilon \cap q_\delta$, if not empty, is an interval s_η whose upper bound $s + \eta$ is either $p + \varepsilon$ or $q + \delta$, and thus the condition $a < \iota(s + \eta)$ holds for all $a \in \mathcal{A}$ automatically since it holds for both $p + \varepsilon$ and $q + \delta$. Hence, to conclude that $s_\eta \in \mathcal{B}$ it suffices to show that $p_\varepsilon \in a''$, since then $p_\varepsilon \cap q_\delta \in a''$ too, and is thus non-empty. With the aid of Proposition 4.12 we have $\iota(p - \varepsilon) < a'$, and we wish to show that $\iota(p - \varepsilon) < a'' < \iota(p + \varepsilon)$. But $a'' < \iota(p + \varepsilon)$ is immediate from $p_\varepsilon \in \mathcal{B}$, while $\iota(p - \varepsilon) < a' < a''$ follows from the preceding inequalities.

To show that \mathcal{B} is Cauchy let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{2}$. For each $a \in \mathcal{A}$ we may find $p(a) \in \mathbb{Q}$ such that $p(a)_\delta \in a$, and in particular $\iota(p(a) + \delta) > a$. It suffices to exhibit a single $a_0 \in \mathcal{A}$ for which $\iota(p(a_0) + \varepsilon) > a$, for all $a \in \mathcal{A}$, since then (as $p(a_0)_\varepsilon \supseteq p(a_0)_\delta$) $p(a_0)_\varepsilon \in a_0$ and thus $p(a_0)_\varepsilon \in \mathcal{B}$, as required for the Cauchy condition. Suppose to the contrary that for all $a \in \mathcal{A}$ there exists $a' \in \mathcal{A}$ with $a' \geq \iota(p(a) + \varepsilon)$. In particular, $a' \geq \iota(p(a) + 2\delta) > a + \iota(\delta)$. Starting with an arbitrary $a_0 \in \mathcal{A}$, an inductive argument then shows that for arbitrary $n \in \mathbb{N}$ an element $a \in \mathcal{A}$ exists with $a > a_0 + \iota(n\delta)$. But as b is an upper bound of \mathcal{A} it follows that $b \geq a_0 + \iota(n\delta)$, for all $n \geq 1$, contradicting the fact that \mathbb{R} is archimedean.

5. Consequences

In this final section we inspect two aspects of the real numbers through the lens of the above formalism; the uncountability of the reals and the limit definition. The criterion for equality of real numbers plays a crucial role.

Proofs of the uncountability of the reals are in (relative) abundance. Typically, such a proof relies either on a particular representation of real numbers (e.g., Cantor's famous diagonalization proof on the digits of expansions of the real numbers) or on some (more or less immediate) properties of the real number system (e.g., the reals are uncountable since \mathbb{R} is a non-empty complete metric space, applying Baire's category theorem).

The construction of the reals above lends itself straightforwardly to yet another proof which uses only basic properties of filters and the criterion for equality (Theorem 4.4). We first observe a trivial auxiliary result. For rational intervals I and J let $|I|$ denote the length of the interval, and we write $J \subseteq_d I$ if J is *deeply contained* in I , meaning, assuming $I = (p, q)$, that there exists $\varepsilon > 0$ such that $J \subseteq (p + \varepsilon, q - \varepsilon)$. Then given a real number a and a rational interval I , there exists a rational interval $J \subseteq_d I$ with $|J| = |I|/5$, and $A \in a$ with $J \cap A = \emptyset$. Indeed, since a is Cauchy, there exists some interval $A \in a$ with $|A| \leq |I|/5$. Subdividing

I into five equal intervals, any of the middle three is deeply contained in I , and A can not have non-empty intersection with all three.

Theorem 5.1. *The set \mathbb{R} is uncountable.*

Proof. Given a sequence $(a_n)_{n \geq 1}$ of real numbers it suffices to construct a real number not in the sequence. Choose an arbitrary rational interval I_0 . Suppose that a sequence $I_n \subseteq_d I_{n-1} \subseteq_d \cdots \subseteq_d I_0$ was constructed together with A_1, \dots, A_n , where $A_k \in a_k$, $1 \leq k \leq n$, and with the properties that $|I_{k+1}| = |I_k|/5$ for all $0 \leq k < n$, and such that $I_n \cap A_n = \emptyset$ for all $1 \leq k \leq n$. Considering the real number a_{n+1} and the interval I_n we may find a rational interval $I_{n+1} \subseteq_d I_n$ with $|I_{n+1}| = |I_n|/5$ and $A_{n+1} \in a_{n+1}$ so that $I_{n+1} \cap A_n = \emptyset$. Continuing in this fashion, the sequence $\{I_n\}_{n \geq 0}$ is a Cauchy sequence of rational intervals which forms a round Cauchy filter base, thus generating a real number a . Since $I_n \in a$ and $I_n \cap A_n = \emptyset$, we conclude that $a \neq a_n$, for all $n \geq 1$. \square

A pleasant consequence of the filters formalism is the following reformulation of the definition of convergence of a sequence of real numbers. The lack of an explicit appearance of ε should be noted.

Lemma 5.2. *A sequence $(a_n)_{n \geq 1}$ of real numbers converges to a real number b if, and only if, for all $B \in b$ there exists $n_0 \in \mathbb{N}$ such that $B \in a_n$, for all $n \geq n_0$.*

Proof. Assume the condition holds. Given $\varepsilon > 0$ consider $q \in \mathbb{Q}$ with $q_{\varepsilon/2} \in b$, which is equivalent to $\iota(q - \varepsilon/2) < b < \iota(q + \varepsilon/2)$. Let $n_0 \in \mathbb{N}$ be given by the assumed condition. Then, for each $n \geq n_0$, $q_{\varepsilon/2} \in a_n$ and so $\iota(q - \varepsilon/2) < a_n < \iota(q + \varepsilon/2)$. Together with the estimate for b we obtain $b - \iota(\varepsilon) < a_n < b + \iota(\varepsilon)$, the familiar convergence formulation.

Conversely, suppose that $a_n \rightarrow b$ in the usual sense, and let $B \in b$. By roundness and the Cauchy condition we may then find $q \in \mathbb{Q}$ and $\varepsilon > 0$ with $q_{\varepsilon/2} \in b$ and $q_\varepsilon \subseteq B$. In particular, $\iota(q - \varepsilon/2) < b < \iota(q + \varepsilon/2)$. Let $n_0 \in \mathbb{N}$ be such that $b - \iota(\varepsilon/2) < a_n < b + \iota(\varepsilon/2)$, for all $n \geq n_0$. Then $q - \iota(\varepsilon) < a_n < q + \iota(\varepsilon)$, and thus $q_\varepsilon \in a_n$, for all $n \geq n_0$. Since $q_\varepsilon \subseteq B$ it follows that $B \in a_n$, for all $n \geq n_0$, as required. \square

Note that one may now establish the uniqueness of limits as follows. If b, c are both limits of the sequence $(a_n)_{n \geq 1}$, then, for all $B \in b$ and $C \in c$, there exists $n_0 \in \mathbb{N}$ with, in particular, $B, C \in a_n$. Since a_n is a proper filter, it follows that $B \cap C \neq \emptyset$, and thus $b = c$ by the equality criterion for real numbers.

Much of the fundamentals of analysis can effectively be developed along similar lines. For instance we mention the following. Consider a sequence $(a_n)_{n \geq 1}$ of real numbers. Let \hat{a} be the collection of all subsets $\hat{A} \subseteq \mathbb{Q}$ for which there exists $n_0 \in \mathbb{N}$ with $\hat{A} \in a_n$, for all $n \geq n_0$. It is a trivial matter to verify that \hat{a} is a proper filter. We leave the details of the following claim for the amusement of the reader. Let us denote by \mathcal{F}_∞ the filter generated by $\{(q, \infty) \subseteq \mathbb{Q} \mid q \in \mathbb{Q}\}$, with $\mathcal{F}_{-\infty}$ defined similarly.

Theorem 5.3. *Let $(a_n)_{n \geq 1}$ and \hat{a} be as above. Then*

- (1) $(a_n)_{n \geq 1}$ converges $\iff \hat{a}$ is a Cauchy filter $\iff (a_n)$ is a Cauchy sequence.
- (2) if $(a_n)_{n \geq 1}$ converges, then it converges to $(\hat{a})_\circ$.
- (3) $c \in \mathbb{R}$ is a partial limit of $(a_n)_{n \geq 1}$ $\iff c \supseteq \hat{a}$.

(4) ∞ is a partial limit of $(a_n)_{n \geq 1} \iff \mathcal{F}_\infty \supseteq \hat{a}$ (and similarly for $-\infty$).

Our presentation of the construction of the reals has come to an end. We hope the reader enjoyed its geometric flavour and the inherent strong appeal to rational approximations, and hopefully she found the proofs and progression elegant. We conclude with the pedagogical remark that the techniques one learns in the course of the construction (namely getting acquainted with filters) are useful in topology and in analysis and are not ad-hoc tools just for this construction. Moreover, the rather clean convergence criterion obtained in this section may indicate that this particular construction of the real numbers can serve as a firm bridge toward the study of elementary analysis rather than being a swamp of convoluted details one never wants to see again (or ever).

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