Constructing the hereditary crossed product order containing a given weak crossed product order and a criterion for weakness

by Christopher James Wilson

The author considers the following set up: R is a DVR with quotient field F and K is a finite cyclic extension of F with group $G = \langle\sigma\rangle$, $f : G \times G \mapsto K - \{0\}$ is a normalised 2-cocycle which satisfies $f(G \times G) \subseteq S$, where S is the integral closure of R in K. It is always assumed that S is a local ring and the extension K/F is a tamely ramified. The crossed product algebra $\Sigma_f = \sum_{\gamma \in G} K x_\gamma$ naturally contains the R-order $C_f = \sum_{\gamma \in G} S x_\gamma$ in this situation. The subgroup H of G consisting of $\{\gamma \in G \mid x_\gamma \text{ is a unit in } C_f\}$ is called the *inertial subgroup* for C_f. Since it is not assumed that $H = G$, C_f is a so-called *weak* crossed product order. Such orders were first introduced in [D. E. Haile, J. Algebra 105 (1987), 116-148. MR0871749 (88b:16013)], without the assumption that S is local, and have since been studied by other authors including this reviewer. Unlike the classical crossed product orders over valuation rings whereby the inertial subgroup is the whole of G, weak crossed product orders are present in all crossed product algebras over valued fields, hence their utility.

Let $v : K \mapsto \mathbb{Z}$ be a valuation of K corresponding to S. Starting with little more than the cocycle f, the inertial subgroup H of C_f, and the integer $\sum_{i=1}^{|G|} v(f(\sigma, \sigma^i))$, the article devises a short, six-step ingenious computational scheme for constructing a hereditary weak crossed product R-order of Σ_f containing C_f. As a bonus, the verification process for the algorithm shows that this is the only hereditary crossed-product order in Σ_f containing C_f, and that it is a classical crossed product order precisely when the integer $\sum_{i=1}^{|G|} v(f(\sigma, \sigma^i))$ is a multiple of $|G|$. The paper concludes by briefly outlining a procedure that can be used for constructing the hereditary crossed product order containing C_f in non-cyclic algebras as well, and how to tell if such an R-order will turn out to be a classical crossed product order.

In addition to the procedures mentioned above, the article characterizes hereditary weak crossed product R-orders in not necessarily cyclic algebras as those R-orders that are maximal with respect to inclusion among the weak crossed product R-orders in Σ_f, as long as S is a local ring. When one drops the assumption that S is local, this may no longer hold. (The hereditary weak crossed product order given as an example in [J. S. Kauta, Proc. Amer. Math. Soc. 141 (2013), 1545-1549] is contained in two maximal orders, each one a crossed product order by Proposition 1.3 of Haile's paper [op. cit.]).