A Goal Programming Approach: Multi-objective Optimization

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Abstract

The purpose of this paper is to accentuate the development of multi-objective non-linear programming (MONLP) technique and its advantages of applying to numerical problems. In particular, non-linear programming model is the process of solving an optimization problem defined by a system of inequalities along with an objective function of several variables that exist in various fields. In certain instances, there are situations in these fields where multiple objectives are required to be achieved simultaneously, owing to limited timeframe and convenience of budget. The Multi-objective programming under non-linear conditions and the solution procedure on the goal programming approach is embedded with algorithm and the relevant technique is developed. Numerical examples, specifically, multi-objective quadratic programming problem and examples of other multi-objective non-linear programming problem are presented to illustrate practical use and the computational details of the proposed procedure. The proposed goal programming technique is then solved using a user-friendly optimization software LINGO.

Keywords: Multi-objective non-linear programming, solution procedure, Goal programming, multi-objective quadratic programming, LINGO

1. INTRODUCTION

Real life situations consist of problems some of which are non-linear in nature. Decision makers may need to solve multiple dependent objectives or criteria in that decision makers have increased recognitions that most real-life decision problems characteristically consist of multiple objectives. These decision-making problems with multiple linear and non-linear objectives or criteria are generally known as multi-

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objective non-linear optimization or multi-objective non-linear programming (MONLP) problems. MONLP problem considered in this paper is a multi-objective programming problem that deals with more than one objective function along with if any one of these multiple objectives is non-linear, then the problem is considered as MONLP problem. The set of constraints consist of linear inequalities with nonnegative restrictions on the decision variables. Ascertaining the optimality criteria, a solution procedure on Goal Programming approach is proposed to solve MONLP that develops solution procedures for determining the optimum compromised solution for the problem. The algorithm of the proposed procedure for solving MONLP problem is stated in which case it considers generalized MONLPP with objectives where maximization objectives are converted into minimization. This is followed by certain numerical examples to illustrate the practical utilization and computational details of the proposed solution procedure. The initial example is a simple multi-objective quadratic programming problem (MOQPP) indicating the revenue and investment objective functions with constraint as budget to illustrate the computational details of the proposed goal programming approach for determining the compromise solution of the MOQPP. Non-linear programming (NLP) technique; Beale algorithm is used to solve the objective functions separately until an optimum basic feasible solution is obtained. Further, two additional MONLP problem consisting of NLP problem of two variables reported by Sirinivas and Deb (1994) together with Osyczka and Kundu (1995) is also solved to determine the compromise solution using the goal programming approach. The proposed goal programming procedure is then solved using a user-friendly optimization software; LINGO. The results of the numerical examples that has been solved utilizing the new proposed technique is compared with solution of other techniques to demonstrate the strength of the proposed method.

2.　STUDIES OF TECHNIQUE AND METHOD DEVELOPMENT

Several attempts have been made by several authors in developing the techniques of solving multi-objective non-linear programming problems. Development of an evolutionary algorithm method was initiated by Deb (2001). Alternatively, Khan et al. (2010) proposed a technique of detecting an integer solution of a multi-objective stratified sampling using a goal programming approach. Guddat et al. (2007) proposed the application of path following method with jumps to multi-objective optimization problems for which several computational results were presented. Considerable number of publications have been created in the past and more recently on fuzzy logic. In anticipation, Sakawa and Yano (1989) illustrated an interactive fuzzy satisficing method for multi-objective non-linear programming problems inclusive of fuzzy numbers. Conversely, Sasaki and Gen (2003) proposed a multi-objective non-linear programming method on problems which have fuzzy multiple objective functions and constraints with Generalized Upper Bounding structure that is solved by proposed Hybridized Genetic Algorithms (HGA). Similarly, Li et al. (2008) presented a genetic algorithm (GA) based method to solve fuzzy multi-objective non-linear programming (FMONLP) problem. Sakawa and Kato (2009) focused on MONLPP including block-angular structures. More previously, Farahi and Ansari (2010) used fuzzy programming

to alter the multi-objective linear bi-level programming problem to a linear bi-level problem. Further, enlarged the *k*-th ideal technique to solve the final problem that was considered as a new approach. Moreover, Kanaya (2010) posed an algorithm method that was interactive cutting plane. El-Sobky and Abo-Elnaga (2013) introduced a numerical technique to decide on the stability set of the initial kind for fuzzy multi-objective non-linear programming (FMONLP) problem. Most authors have addressed cases on bi-level programming problems. Emam (2011) proposed an interactive algorithm for solving a bi-level multi-objective integer non-linear programming (BLMINP) problem with linear or non-linear objectives. Apart from the methods discussed above, there are many authors who developed interactive methods pertaining to MONLP problem. For instance, Yano (1999) proposed an interactive algorithm to procure a satisfying solution efficiently from the two-level type M-Pareto optimal (2-MPO) solution. However, Vassilev et al. (2002) presented learning-oriented interactive reference direction algorithm for solving multi-objective convex non-linear integer programming problems. Baky (2013) developed a methodology for solving multi-level non-linear multi-objective decision making (MLN-MODM) problems of maximization type by broadening the concept of the method for order of preference by similarity to ideal solution (TOPSIS). Literature on the development of methods and algorithms for MONLP have also been published in the past and recently. Masud and Zheng (1989) presented an algorithm for resolving multi-objective non-linear programming problem and termed it as direction-searching method. In contrast, Utyuzhnikov et al. (2009) presented a new approach set in non-linear multi-objective optimization for generating a well-distributed Pareto. However, Pandian and Natarajan (2009) introduced a new class of functions namely, second order (b,F)-type I convex functions which is the extension of type I, F-type I and b-type I functions. Writings have also been indicated on Pareto optimal outcome whereby new techniques and algorithms on multi-objective non-linear programming have been introduced. Hartikainen et al. (2011) introduced a technique, Pareto Front Interpolation for Nonlinear Multi-objective Optimization (PAINT). Applications of MONLP to real life situations have been studied by many authors. New algorithms and techniques have been presented and applied to real life situation considering multiple-objective nonlinear programming problems. In contrast, Abo-Sinna and Baky (2007) presented three-level multi-objective decision-making (TL-MODM) models that are non-linear function. Recently, Ozlen (2013) proposed a general algorithm to optimize a non-linear utility of multiple *k*-function of multiple objectives over the integer programming efficient set. Several literatures focus on duality approach considering multi-objective nonlinear programming problem. Huang and Yang (2002) introduced vector valued non-linear Lagrangian functions and non-linear penalty functions for multi-objective constrained optimization problems. Moreover, Mishra and Wang (2005) formulated two pairs of second order symmetric duality for non-linear multi-objective mixed integer programs for arbitrary cones.

3. THE MULTI-OBJECTIVE NON-LINEAR PROGRAMMING (MONLP) PROBLEM

A multi-objective programming is a discipline of mathematical optimization problems for which there exists quantitative objective functions of a system that are to be either minimized or maximized simultaneously. Generally, multi-objective linear programming is a mathematical representation of the objective in terms of a measurable quantity. Initially, MOLP problem has p objective functions of n decision variables, out of which k are to be maximized and $p-k$ are to be minimized subject to m constraints on $\mathbb{R}^n$.

Therefore, the objectives are $z_1, z_2, \ldots, z_k, \ldots, z_p$, the number of objective functions is $= p$, $k = 1, \ldots, p$, the variables are $x_1, x_2, \ldots, x_j, \ldots, x_n$ and the number of variable functions are $= n$, $j = 1, 2, \ldots, n$.

$$\text{Maximize(or Minimize)} \ \ f_1(x_1, x_2, \ldots, x_n), \ldots, f_p(x_1, x_2, \ldots, x_n) \tag{1.1}$$

$$\text{subject to} \quad g_1(x_1, x_2, \ldots, x_n) \quad (\leq, = \text{ or } \geq) \quad b_1$$
$$\vdots \qquad\qquad \vdots$$
$$g_m(x_1, x_2, \ldots, x_n) \quad (\leq, = \text{ or } \geq) \quad b_m \tag{1.2}$$

$$\text{and} \qquad\qquad x_j \geq 0 \tag{1.3}$$

where the functions, $g_i (i = 1, 2, \ldots, m)$ in equation (1.2) are considered as constraints of the problem, and $(b_1, b_2, \ldots, b_m)$ is a set of constants and the equation (1.3) represents the non-negativity restrictions on the variables $x_j (j = 1, 2, \ldots, n)$. If at least one f_k and/or g_i are non-linear, the problem is considered as multi-objective non-linear programming problem (MONLPP). In essence many real-life problems have both linear and non-linear objective functions.

3.1   THE PROBLEM OF MULTI-OBJECTIVE PROGRAMMING UNDER NON-LINEAR CONDITIONS

Let the Multi-objective Non-linear Programming (MONLP) Problem be given as:

Minimize $\quad \left[f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_p(x_1, x_2, \ldots, x_n) \right]$

subject to $\quad g_i(x_1, x_2, \ldots, x_n) \leq \{\text{or} \ = \ \text{or} \ \geq\} b_i; \quad i = 1, 2, \ldots, m$

and $\quad x_1, x_2, \ldots, x_n \geq 0 \tag{1.4}$

where, at least one of the functions of $f_j(x_1, x_2, ..., x_n)$; $j = 1, 2, ..., p$ and

$g_i(x_1, x_2, ..., x_n)$; $i = 1, 2, ..., m$ is non-linear.

3.2 THE SOLUTION PROCEDURE: A GOAL PROGRAMMING APPROACH

In this section we develop a solution procedure to solve the MONLPP (1.4) using a goal programming technique.

Let $\left(x_{1j}^*, x_{2j}^*, ..., x_{nj}^*\right)$ denote the optimum solution for the jth objective ($j = 1, 2, ..., p$), that is, $\left(x_{1j}^*, x_{2j}^*, ..., x_{nj}^*\right)$ is the solution to the following NLPP:

Minimize $f_j(x_1, x_2, ..., x_n)$; $j = 1, 2, ..., p$

subject to $g_i(x_1, x_2, ..., x_n) \leq \{\text{or } = \text{ or } \geq\} b_i$; $i = 1, 2, ..., m$,

and $x_1, x_2, ..., x_n \geq 0$ (1.5)

Let f_j^* be the optimum value of the objective function of (1.5) at $\left(x_{1j}^*, x_{2j}^*, ..., x_{nj}^*\right)$.

Further, let $\left(x_1^*, x_2^*, ..., x_n^*\right)$ be the optimum compromise solution, that is, $\left(x_1^*, x_2^*, ..., x_n^*\right)$ is the solution to the MONLPP (1.4) with f_j as the value of the jth objective function.

Obviously, $\left(x_1^*, x_2^*, ..., x_n^*\right)$ incurs a loss in f_j due to not utilizing its individual optimum solution $\left(x_{1j}^*, x_{2j}^*, ..., x_{nj}^*\right)$. That is, $f_j \geq f_j^*$.

Let $z_j = f_j - f_j^*$.

Then, the quantity $z_j \geq 0$; ($j = 1, 2, ..., p$) denotes the maximum loss in jth objective owing to the utilization of compromise solution $\left(x_1^*, x_2^*, ..., x_n^*\right)$ instead of using its individual optimum solution $\left(x_{1j}^*, x_{2j}^*, ..., x_{nj}^*\right)$.

Thus, solve the MONLP Problem (1.4) by setting the following "Goal":

"Find such compromise solution $\left(x_1^*, x_2^*, ..., x_n^*\right)$ for which the loss in the value of the jth objective function in MONLPP (1.4) should not exceed the quantity z_j ".

Therefore, $\left(x_1^*, x_2^*, ..., x_n^*\right)$ must satisfy:

$$f_j - f_j^* \leq z_j,$$

or $f_j - z_j \leq f_j^*$ (1.6)

Introducing (1.6) as an additional constraint to each objective, a reasonable criterion to determine the optimum compromised solution $\left(x_1^*, x_2^*, ..., x_n^*\right)$ to MONLPP (1.4) can be worked out to minimize the sum of the loss in all the p objectives. That is,

$$\text{Minimize} \qquad \sum_{j=1}^{p} z_j \qquad\qquad\qquad (1.7)$$

Therefore, incorporating the constraint in (1.6) and the objective given in (1.7), the equivalent problem to the MONLP problem (1.4) may be expressed as the following goal programming problem (GPP):

Minimize $\displaystyle\sum_{j=1}^{p} z_j$

subject to $f_j - z_j \leq f_j^*, \quad j = 1, 2, ..., p$

$$g_i\left(x_1, x_2, ..., x_n\right) \leq \{\text{or} = \text{or} \geq\} b_i; \quad i = 1, 2, ..., m,$$

and $x_1, x_2, ..., x_n \geq 0$ (1.8)

The Goal Programming Problem in (1.8) may be solved by executing a program utilizing LINGO software package for solving non-linear, linear and integer optimization problems developed by LINDO Systems, Inc.

3.3   THE ALGORITHM

The algorithm of the proposed solution procedure discussed in Section 3.2 for solving the MONLPP given in (1.4) is summarized as follows:

Step-1: Consider MONLPP with p objectives, initial step is to transform all objective functions specifically of maximization   sort possibly if any, into minimization.

Step-2: Set up p separate to NLPP with the purpose to acquire the same set of original

constraints for each objective. An optimal solution f_j^* for all $j = 1, 2, ..., p.$

Step-3: Utilize an appropriate programming technique, specifically non-linear such as Wolfe's method, Kuhn-Tucker conditions, Beale's method, etc. or optimization software for instance LINGO to solve each NLPP independently with the objective f_j constructed in Step-2. Let $f_j^*; j = 1, 2, ..., p$ be assigned as the optimum value of the jth NLPP at its individual optimum solution $\left(x_{1j}^*, x_{2j}^*, ..., x_{nj}^* \right).$

Step-4: If the optimum compromised solution of the given MONLPP is $\left(x_1^*, x_2^*, ..., x_n^* \right),$ that incurs a z_j loss to f_j, then decide on the compromised solution by reforming MONLPP as a GPP with the objective:

$$\text{Minimize} \qquad \sum_{j=1}^{p} z_j$$

subject to the additional constraints $f_j - z_j \le f_j^*; j = 1, 2, ..., p$.

Step-5: Solve the GPP constructed in Step-4 by LINGO to determine the compromise optimum solution $\left(x_1^*, x_2^*, ..., x_n^* \right).$

4   NUMERICAL EXAMPLES OF MONLP PROBLEM

4.1   NUMERICAL EXAMPLES

To illustrate the practical utilization and the computational details of the proposed solution procedure, three numerical examples are presented in Section 4. The Example 4.1 is a non-linear multi-objective quadratic programming problem while the examples 4.2 and 4.3 are the two MONLPPs reported in Osyczka and Kundu (1995) and Srinivas and Deb (1994), respectively.

4.1.1   MULTI-OBJECTIVE QUADRATIC PROGRAMMING PROBLEM

Example 4.1

Apparently, a company is carefully considering to promote product via two media campaigns: television and radio. The costs for company are $2,000 per minute to advertise on television and $1,000 per minute to advertise on radio. The company is willing to spend no more than $4,000. If the company buys x_1 minutes of radio advertising and x_2 minutes of television advertising, then the company is expecting to earn the revenue (in thousand dollars):

$$f_1(x_1, x_2) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

with all possible investment (in thousand dollars):

$$f_2(x_1, x_2) = 10 - 6x_1 + 2x_1^2 - 2x_1x_2 + 3x_2^2.$$

With a minimum investment, how can the company maximize the revenue?

This problem can be solved by formulating the above function as the following Non-linear multi-objective quadratic programming problem (MOQPP):

Maximize	$f_1(x_1, x_2) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$	(Revenue)
Minimize	$f_2(x_1, x_2) = 10 - 6x_1 + 2x_1^2 - 2x_1x_2 + 3x_2^2$	(Investment)
subject to	$x_1 + 2x_2 \le 4$	(Budget)
and	$x_1, x_2 \ge 0$	(1.9)

To demonstrate the computational details of the proposed goal programming approach discussed in Sections 2.2 and 2.3 for determining the optimum compromise solution of the MOQP problem (1.9), the following two quadratic programming problems (QPP) are solved separately to optimize each objective function:

QPP-1:

> Minimize $\quad f_1(x_1, x_2) = -4x_1 - 6x_2 + x_1^2 + 3x_2^2$
>
> subject to $\quad x_1 + 2x_2 \le 4$
>
> and $\quad x_1, x_2 \ge 0$

QPP-2:

> Minimize $\quad f_2(x_1, x_2) = 10 - 6x_1 + 2x_1^2 - 2x_1x_2 + 3x_2^2$
>
> subject to $\quad x_1 + 2x_2 \le 4$
>
> and $\quad x_1, x_2 \ge 0$

The QPP-1 and QPP-2 are then solved using Beale (1959) method as discussed below:

Beale's Algorithm for QPP:

General QPP to consider:

Minimize $f(x) = c'x + \frac{1}{2}x'Qx$

subject to $Ax \{\leq, =, \text{ or } \geq\} b$

and $\quad x \geq 0$

where $x \in \mathbb{R}^n$, A is $m \times n$, b is $m \times 1$, c is $m \times 1$ and Q is $n \times n$ symmetric matrix.

The iterative procedure of Beale' method can be summarized as follows:

Step-1: Convert the maximization $f(x)$, if any, into that of minimization. Introduce slack and/or surplus variables in the inequality constraint to convert it into equality.

Step-2: Choose arbitrarily any m variables, so that remaining $(n-m)$ variables become non-basic. Let the basic and non-basic variables be denoted respectively by:

$$x_B = \left(x_{B_1}, x_{B_2}, ..., x_{B_m}\right) \text{ and } x_{NB} = \left(x_{NB_1}, x_{NB_2}, ..., x_{NB_m}\right)$$

Step-3: Express each basic variable x_{B_i} in terms of non-basic variables x_{NB_i}.

Step-4: Express the objective function $f(x)$ in terms of non-basic variables x_{NB_i}.

Step-5: Evaluate the partial derivatives of $f(x)$ with respect to the non-basic variables at $x_{NB} = 0$ and examine the nature of $\left(\dfrac{\partial f(x)}{\partial x_{NB_k}}\right)_{x_{NB}=0} = 0; k = 1, 2, ..., (n-m):$

(i) If $\left(\dfrac{\partial f(x)}{\partial x_{NB_k}}\right)_{x_{NB}=0} = 0$ for each $k = 1, 2, ..., (n-m)$ and

$\left(\dfrac{\partial f(x)}{\partial x_{u_i}}\right)_{x_{NB}=0, u_i=0} = 0$ for each i, where u_i is a free variable. Then, the current solution is optimum.

(ii) If $\left(\dfrac{\partial f(x)}{\partial x_{NB_k}}\right)_{x_{NB}=0} < 0$ for at least one $k = 1, 2, ..., (n-m)$, the current solution is not optimum.

The non-basic variable corresponding to the most negative of the partial derivatives will enter the basis.

Step-6: Let x_r be the entering variable identified in Step-5. Then, determine the value of x_r by calculating:

(a) The value of x_r that can be attained without driving the present variable (say, x_i) negative.

(b) That the value of x_r at which $\dfrac{\partial f(\boldsymbol{x})}{\partial x_{NB_r}}$ vanishes or equal to zero.

Step-7: Choose the minimum of values obtained in (a) and (b) above.

(i) If it occurs for (a), x_i will be leave the basis.

(ii) If it occurs for (b), none of the basic variables is removed. In such cases, we enlarge x_{NB} by introducing a new non-basic and unrestricted variable u_r, called a free variable, defined by:

$$u_r = \frac{1}{2}\frac{\partial f}{\partial x_r} \quad \text{which becomes an additional constraint.}$$

Step-8: Go to Step-3 and repeat the procedure until an optimum basic feasible solution is obtained after setting $x_{NB} = 0$ and $u_i = 0$.

Solution to QPP-1:

Introducing a slack variable x_3 the QPP-1 can be expressed as:

Minimize $f_1(x_1, x_2) = -4x_1 - 6x_2 + x_1^2 + 3x_2^2$

subject to $x_1 + 2x_2 + x_3 = 4$

    and $x_1, x_2 \geq 0$

Using the Beale's method discussed in steps 1-8, let $x_3 = 4$ be the initial basic variable, Then the basic and non-basic variables are:

$$\boldsymbol{x}_B = \left(x_3\right) \text{and} \quad \boldsymbol{x}_{NB} = \left(x_1, x_2\right)$$

Expressing x_3 and f_1 in terms of non-basic variables x_1 and x_2, we get:

$$x_3 = 4 - x_1 - 2x_2$$

$$\text{and} \quad f_1(x_1, x_2) = -4x_1 - 6x_2 + x_1^2 + 3x_2^2$$

Now, $\left. \dfrac{\partial f_1}{\partial x_1} \right|_{\mathbf{x}_{NB}=0} = 0 \quad \Rightarrow \quad (-4 + 2x_1)_{x_1 = x_2 = 0} = -4$

and $\left. \dfrac{\partial f_1}{\partial x_2} \right|_{\mathbf{x}_{NB}=0} = 0 \quad \Rightarrow \quad (-6 + 6x_2)_{x_1 = x_2 = 0} = -6$

Thus, the non-basic variable x_2, which is corresponded to the most negative of the partial derivatives, will enter the basis. To determine the leaving variable, compute the largest value that x_2 can assume as:

(a) Since $x_3 = 4 - x_1 - 2x_2$ by setting $x_3 = 0$ and $x_1 = 0$, the largest possible

value of $x_2 = 2$

(b) If $\dfrac{\partial f_1}{\partial x_2} = 0$, then $x_2 = 1$

Since the minimum value of x_2 occurs in (b), no basic variable will be removed. We introduce a new non-basic variable:

$$u_1 = \frac{1}{2} \frac{\partial f_1}{\partial x_2} = \frac{1}{2}(-6 + 6x_2) = -3 + 3x_2$$

Adding the above equation as a constraint, we have the new set of basic and non-basic variables:

$$\mathbf{x}_B = (x_3, x_2) \quad \text{and} \quad \mathbf{x}_{NB} = (x_1, u_1)$$

Repeating the steps, it is found that x_1 enters the basis and x_3 leaves from the basis. The sets of basic and non-basic variables are:

$$\boldsymbol{x}_B = (x_1, x_2) \quad \text{and} \quad \boldsymbol{x}_{NB} = (x_3, u_1)$$

Expressing x_1, x_2 and f_1 in terms of non-basic variables x_3 and u_1, we get:

$$x_1 = 2 - x_3 - \frac{2}{3}u_1,$$

$$x_2 = 1 + \frac{1}{3}u_1,$$

and $\quad f_1(x_1, x_2) = -7 + x_3^2 + \frac{4}{9}u_1^2 + \frac{4}{3}u_1 x_3$

For the above f_1 it can be seen that

$$\frac{\partial f_1}{\partial x_3}\bigg|_{x_3 = u_1 = 0} = 0 \quad \text{and} \quad \frac{\partial f_1}{\partial u_1}\bigg|_{x_3 = u_1 = 0} = 0$$

Hence, the optimum solution to the QPP-1 is obtained as:

$x_1^* = 2$ and $x_2^* = 1$ with minimum $f_1^* = -7$ by setting non-basic variables u and x_3 to zeros.

Solution to QPP-2:
In a similar manner, the optimum solution to the QPP-2 is obtained as:

$$x_1^* = \frac{9}{5} \quad \text{and} \quad x_2^* = \frac{3}{5} \quad \text{with minimum} \quad f_2^* = \frac{23}{5}$$

Solution to MOQPP (1.9):
To solve the MOQPP in (1.9), the GPP is formulated as follows:

Minimize $\quad z_1 + z_2$

subject to $\quad -4x_1 - 6x_2 + x_1^2 + 3x_2^2 - z_1 \le -7$

$$10 - 6x_1 + 2x_1^2 - 2x_1 x_2 + 3x_2^2 - z_2 \le \frac{23}{5}$$

$$x_1 + 2x_2 \le 4$$

and $\quad x_1, x_2 \ge 0$

Then, the optimum compromise solution $\left(x_1^*, x_2^*\right)$ to the above GPP and hence the MONLPP (1.9) is obtained using LINGO (see LINGO code in Appendix A) as shown below:

$$x_1^* = 1.941176 \text{ and } x_2^* = 0.8235294 \text{ with } z_2^* = 0.8235294$$

With this solution, the optimum of the two objective functions is obtained as:

$$f_1 = 6.9031141 \text{ and } f_2 = 4.7266435$$

4.1.2 OTHER MULTI-OBJECTIVE NON-LINEAR PROGRAMMING PROBLEM

Example 4.2:

Th Non-linear Programming problem of two variables is considered, as reported by Srinivas and Deb (1994) who acquired from Chankong and Haimes (1983), with Non-linear objective functions and linear/non-linear constraints:

Minimize $\quad f_1(x_1, x_2) = 2 + (x_1 - 2)^2 + (x_2 - 1)^2,$ and

Minimize $\quad f_2(x_1, x_2) = 9x_1 - (x_2 - 1)^2$

subject to $\quad x_1^2 + x_2^2 \leq 225$

$$x_1 - 3x_2 + 10 \leq 0$$

$$x_1 \geq -20$$

$$x_1 \leq 20$$

$$x_2 \geq -20$$

$$x_2 \leq 20 \tag{1.10}$$

In order to formulate the compromise solution to (1.10) utilizing the goal programming approach as discussed in Section 1.6, the two NLPPs as stated in (1.10) is solved for each individual objective f_j, $j = 1$ and 2.

The program coded in LINGO (see Appendix B) is executed and the individual optimum solutions to the two NLPPs for the problem (1.10) are found to be:

For NLPP-1: $\left(x_1^*, x_2^*\right) = (1.1, 3.7)$ with minimum $f_1^* = 10.1$.

For NLPP-2: $\left(x_1^*, x_2^*\right) = (0, 15)$ with minimum $f_2^* = -196.0$.

Then, the GPP for MONLP problem in (1.10) can be expressed to determine the compromise solution as:

Minimize $z_1 + z_2$

subject to $2 + (x_1 - 2)^2 + (x_2 - 1)^2 - z_1 \leq 10.1$

$$9x_1 - (x_2 - 1)^2 - z_2 \leq -196.0$$

$$x_1^2 + x_2^2 \leq 225$$

$$x_1 - 3x_2 + 10 \leq 0$$

$$x_1 \geq -20$$

$$x_1 \leq 20$$

$$x_2 \geq -20$$

and $x_2 \leq 20$

Solving the above GPP using LINGO, coded in Appendix B, the optimum compromise solution to the given MONLPP (1.10) is obtained as:

$$\left(x_1^*, x_2^*\right) = \left(0, \frac{10}{3}\right) \text{ with } z^* = 191.9$$

Example 4.3:

Consider the following NLPP with two objective functions and six variables as reported in Osyczka and Kundu (1995):

Minimize $f_1(x_1, x_2, ..., x_6) = -[25(x_1 - 2)^2 + (x_2 - 2)^2 + (x_3 - 1)^2 + (x_4 - 4)^2 + (x_5 - 1)^2]$, and

Minimize $f_2(x_1, x_2, ..., x_6) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$

subject to $x_1 + x_2 - 2 \geq 0$

$$6 - x_1 - x_2 \geq 0$$

$$2 - x_1 + x_2 \geq 0$$

$$2 - x_1 + 3x_2 \geq 0$$

$$4 - (x_3 - 3)(x_3 - 3) - x_4 \geq 0$$

$$(x_5 - 3)(x_5 - 3) + x_6 - 4 \geq 0$$

$$x_1 \leq 10, x_2 \leq 10, x_6 \leq 10, x_3 \geq 1, x_5 \geq 1, x_3 \leq 5, x_5 \leq 5, x_4 \leq 6$$

$$x_1, x_2, x_4, x_6 \geq 0 \tag{1.11}$$

To formulate and calculate compromise solution to (1.11) using the proposed goal programming approach, the two NLPPs as stated in (1.5) is solved for each individual objective f_j.

Using LINGO program (see LINGO code in Appendix C), the individual optimum solution for $j = 1$ and 2 are found to be:

For NLPP-1: $\left(x_1^*, x_2^*, ..., x_6^*\right) = (0, 2, 1, 0, 1, 0)$ with minimum $f_1^* = -116.0$.

For NLPP-2: $\left(x_1^*, x_2^*, ..., x_6^*\right) = (1, 1, 1, 0, 1, 0)$ with minimum $f_2^* = 4.0$.

Then, the GPP for MONLPP in (1.11) can be expressed as:

Minimize $\quad z_1 + z_2$

subject to $\quad -[25(x_1 - 2)^2 + (x_2 - 2)^2 + (x_3 - 1)^2 + (x_4 - 4)^2 + (x_5 - 1)^2] - z_1 \leq -116$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 - z_2 \leq 4$$

$$x_1 + x_2 - 2 \geq 0$$

$$6 - x_1 - x_2 \geq 0$$

$$2 - x_1 + x_2 \geq 0$$

$$2 - x_1 + 3x_2 \geq 0$$

$$4 - (x_3 - 3)(x_3 - 3) - x_4 \geq 0$$

$$(x_5 - 3)(x_5 - 3) + x_6 - 4 \geq 0$$

$$x_1 \leq 10, x_2 \leq 10, x_6 \leq 10, x_3 \geq 1, x_5 \geq 1, x_3 \leq 5, x_5 \leq 5, x_4 \leq 6$$

$$x_1, x_2, x_4, x_6 \geq 0$$

By solving the above GPP using LINGO as coded in Appendix C, the optimum compromise solution to the given MONLPP (1.11) is obtained as:

$$\left(x_1^*, x_2^*, ..., x_6^*\right) = \left(0, 2, 1, 0, 1, 0\right) \text{ with } z^* = 2.0$$

5 CONCLUSION

Multi-objective optimization is a crucial topic in non-linear programming problems. This technique is widely utilized in many real-life problems as the decision makers often encounter the optimizing simultaneously several functions. There are several techniques available in writing which deal with multi-objective non-linear programming problem. Fuzzy goal programming and ε-constraint method are also useful multi-objective optimization techniques as there are various possibilities for measuring the fuzziness around the target goals, each of which leads to a different fuzzy membership function. Furthermore, the membership function requires transformation into membership goals by assigning the highest degree as the aspirational level and introducing under- and over- deviational variables. This approach is lengthy process and has a major drawback owing to additional over- and under- deviational variables. Alternatively, the ε-constraint method is developed for general multi-objective problems. The solution procured largely depends on the chosen ε vector. Furthermore, the ε vector must lie within and the solution procedure is more strenuous to implement. The major setback in applying some of these methods is that if the number of objective functions is large then the computational effort required to generate an efficient set of solutions is prohibitive. Moreover, there is no guarantee of detecting optimal solution in a certain amount of time. Another drawback is that the population tends to converge to solutions that are superior in one objective function but not in the other objective functions. A solution procedure utilizing goal programming approach is developed for solving MONLPP in this article. This new technique is based on the condition which does not allow an increase in the value of jth objective (minimize) function more than

a quantity (z_j), where z_j is the maximum increase that may occur due to utilization of compromised solution, instead of using its individual optimum solution. This new

technique is based on the condition which does not allow to increase in the value of jth objective (minimize) function more than a quantity z_j where the maximum increase may occur due to the use of compromise solution, instead of using its individual optimum solution. The optimum compromised solution is worked out by formulating the problem as an MONLP that seeks minimization of the sum of the increase in the objective function of all the objectives. The formulated MONLPP is solved by LINGO. Numerical examples illustrate the computational details. The benefits of the developed techniques are that they will be applicable and can be utilized to nearly all real-life situations, even with non-linear multi-criteria conditions. Beale's algorithm is stated with the solution method for multi-objective quadratic programming problems. Other MONLP problems are also stated to work out the compromise solution using the goal programming approach. Some of the limitations are that real-life practical interests have hundreds of variables with decimal as quantitative variables. Thus, formulation of the problem may become difficult, therefore, computation of the values will have to be done utilizing mathematical software. Reasonable goals and targets of the MOLP and MONLP problems cannot be specified without reference to underlying decision-maker preferences and thus makes the formulation of the problems difficult.

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DATA AVAILABILITY

The data that support the findings of this study are available from the first author, Nirma Narayan, upon reasonable request.

DISCLOSURE STATEMENT

No potential conflict was reported by the author(s).

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9   APPENDIX

Appendix A:

LINGO Code for Example 4.1

```
min=z1+z2;
-4*x1-6*x2+x1^2+3*x2^2-z1<=-7;
10-6*x1+2*x1^2-2*x1*x2+3*x2^2-z2<=4;
x1+2*x2<=4;
```

Appendix B:

LINGO Code for Example 4.2

For NLPP-1:

Min=2+(x1-2)^2+(x2-1)^2;

x1^2+x2^2<=225;

x1-3*x2+10<=0;

x1>=-20;

x1<=20;

x2>=-20;

x2<=20;

For NLPP-2:

Min=9*x1-(x2-1)^2;

x1^2+x2^2<=225;

x1-3*x2+10<=0;

x1>=-20;

x1<=20;

x2>=-20;

x2<=20;

For GPP:

Min=z1+z2;

2+(x1-2)^2+(x2-1)^2-z1<=10.1;

9*x1-(x2-1)^2-z2<=-196;

x1^2+x2^2<=225;

x1-3*x2+10<=0;

x1>=-20;

x1<=20;

x2>=-20;

x2<=20;

Appendix C:

LINGO Code for Example 4.3

For NLPP-1:

```
Min=-(25*(x1-2)^2+(x2-2)^2+(x3-1)^2+(x4-4)^2+(x5-1)^2);
x1+x2-2>=0;
6-x1-x2>=0;
2-x2+x1>=0;
2-x1+3*x2>=0;
4-(x3-3)^2-x4>=0;
(x5-3)^2+x6-4>=0;
x1>=0;x2>=0;x6>=0;
x1<=10;x2<=10;x6<=10;
x3>=1;x5>=1;x3<=5;x5<=5;
x4>=0;x4<=6;
```

For NLPP-2:

```
Min=x1^2+x2^2+x3^2+x4^2+x5^2+x6^2;
x1+x2-2>=0;
6-x1-x2>=0;
2-x2+x1>=0;
2-x1+3*x2>=0;
4-(x3-3)^2-x4>=0;
(x5-3)^2+x6-4>=0;
x1>=0;x2>=0;x6>=0;
x1<=10;x2<=10;x6<=10;
x3>=1;x5>=1;x3<=5;x5<=5;
x4>=0;x4<=6;
```

For GPP:

Min=z1+z2;

-(25*(x1-2)^2+(x2-2)^2+(x3-1)^2+(x4-4)^2+(x5-1)^2)-z1<=-116;

x1^2+x2^2+x3^2+x4^2+x5^2+x6^2-z2<=4;

x1+x2-2>=0;

6-x1-x2>=0;

2-x2+x1>=0;

2-x1+3*x2>=0;

4-(x3-3)^2-x4>=0;

(x5-3)^2+x6-4>=0;

x1>=0;x2>=0;x6>=0;

x1<=10;x2<=10;x6<=10;

x3>=1;x5>=1;x3<=5;x5<=5;

x4>=0;x4<=6;