U.Mehta, S. Lal, M. Tauqeer and V. Prasad, "Identifying coupled TITO systems with fractional-order model using block-pulse operational matrix and a single relay closed-loop experiment" in *Fractional Dynamics in Natural Phenomena and Advanced Technologies*, D. Baleanu, J. Hristov, eds., Cambridge Scholars Publishing, 2024, pp.151-165. https://www.cambridgescholars.com/product/978-1-5275-5276-0

CHAPTER 7

IDENTIFYING COUPLED TITO SYSTEMS WITH FRACTIONAL-ORDER MODEL USING BLOCK-PULSE OPERATIONAL MATRIX AND A SINGLE RELAY CLOSED-LOOP EXPERIMENT

UTKAL MEHTA[A], SURAJ LAL[A],
MOHAMMED TAUQEER[A] AND VINEET PRASAD[A]
[A] ELECTRICAL AND ELECTRONIC ENGINEERING, SCHOOL OF INFORMATION TECHNOLOGY, ENGINEERING, MATHEMATICS AND PHYSICS, THE UNIVERSITY OF THE SOUTH PACIFIC, FIJI

Introduction

Identifying multivariable systems has been one of the most challenging tasks for control engineers. The system that is represented by two-input-to-output (TITO) signals with cross-coupling is of great importance in process industries (Nordfedlt, 2005). Over the last decade, several authors have proposed relevant identification procedures for modelling TITO systems. Modelling any system, whether single-input-single-output (SISO), TITO, or multi-input-multi-output (MIMO) systems, has become the fundamental approach to determining the behaviour of the system. The plant or the system is a numerical representation of a mechanical, electrical, or a combination of the two. This numerical representation is recognized as the transfer function of a system with no orders, most commonly known as the integer order transfer function (IOTF). This transfer function is the result of integer order differential equations and the Laplace transform using classical methods. Once the IOTF is achieved, a linear controller is designed to compensate for the desired output in terms of settling time, steady-state error, and more. However, an IOTF would never be able to represent the real-time data of a system exactly due to the natural order of a real-time system in the execution.

A real-time model does not limit the operation to a specific integer-order operator (derivative). However, the real-time systems behave dynamically with time shifts between the input and output signals, and consequently, the fractional calculus techniques are highly invoked. Since fractional calculus allows more general descriptions of dynamical systems, it could be anticipated that such models can provide more accurate and precise descriptions of the dynamic systems compared to the classical integer methods (Tepljakov, 2017).

In recent years, fractional order calculus has attracted engineers due to its ability to describe the behaviour of a dynamic system in compact expressions and its infinite memory characteristics. The primary reason for not using fractional calculation in earlier times was the lack of solution methods for fractional differential equations. Nevertheless, currently, many methods of approximation of fractional derivatives and integrals are in place and being used as promissory tools in bioengineering, viscoelasticity, electronics, robotics, control theory, and signal processing (Kothari, 2019).

In the real world, fractional-order models have proven very effective in describing the behaviour of viscoelastic systems, ocean fishery models, supercapacitors, lithium-ion batteries, virus models, smoking quitting behaviour, human arm dynamics, and atmospheric dynamics of carbon dioxide gas.

Two major methods of fractional order model identification are frequency and time domains.

- The time domain method typically includes the equation error method and output error method. Many researchers have explored strategies for a simplification of fractional order dynamics, strategies that transform fractional order systems into algebraic systems.
- A modulating function was utilized to change the calculation of a signal's input/output fractional derivative. Solutions to utilize the polynomial functions, and Gaussian function, are different from integer order model identification.

Nonetheless, fractional-order systems are greatly challenged by the discretization problem (Li and Sun, 2011). An operational matrix method proposed by Tang et al., (2015) converts the fractional differential or integral operators in an algebraic equation to a generalized operational matrix of block pulse functions. That method does not restrict the identification of fractional orders of the fractional systems to be identified.

As reported by Zhang et al., (2022), a block pulse function is constant and not sufficiently smooth, and thus, a hybrid of the Bernoulli principle and block pulse function was proposed. This can decrease the operational matrix dimensions and computational complexity. The parameter identification for fractional order systems was achieved by reducing the mean square error between the output and true system using the algebraic fractional order system. That was found to be a more accurate approximation when compared to elemental block pulse functions. On the contrary, the hybrid operator matrix involves a complex matrix. Hence, there is a need to develop methods to identify fractional-order systems with time delays.

In the implementation of fractional-order derivatives and integrals, the most frequently utilized definitions are the left and right-sided Riemann-Liouville (RL) integrals and the left and right-sided Caputo and Riesz derivatives (Bo et al., 2021, Mehta et al., 2022). However, Caputo and Riesz derivatives are generally inconsistent with integer-order derivatives. On the contrary, in most cases, fractional models for the identification of Riemann-Liouville, Caputo, and Grunwald-Letnikov (GL) parameters have been used.

The Grunwald-Letnikov derivative is widely used in applications related to digital control because its discretization can be simply fitted to real data (Kothari and Mehta, 2021). Moreover, some recent studies reveal that non-singular fractional calculus derivatives could be used, such as the Caputo-Fabrizio derivative and Atangana-Baleanu derivatives (Fareed et al., 2022). In addition, recent applications of fractional operators to control applications have proven to yield promising results (Ranjan and Mehta, 2022). In the sequel, the Riemann-Liouville derivative is applied to obtain a fractional model of a coupled TITO system.

The literature sources cited above reveal that the identification method of multivariable systems is still a challenging problem due to the strong coupling between two or more cross-coupled subsystems. Such a system can require a multi-step approach and additional decoupling to counter-system interactions. Also, the performance of any dynamic process is highly determined by an adequate model and the art of its build-up.

This chapter provides a schematic to estimate the fractional first-order delay model from simple input and output data, generated from a single experiment. The new fractional modelling scheme does not require any additional sequential steps of identification or decouplers. The orthogonal series-based algebraic technique, namely the block-pulse operational matrix, is used to address real-world engineering applications involving a large amount of system identification data and system noise.

A one-step approach produces faster identification with less experimental time. The feasibility of engineering applications can be demonstrated by digitally examining the examples presented.

Two Input Two Output (TITO) System

Systems having two inputs and two output signals are, in general, cross-coupled and play significant roles in process system dynamics. TITO systems fall under one of the branches of MIMO systems, and when considering the loops separately, they resemble a SISO system. Numerous techniques have been developed for the identification of a TITO system and the design of its controller. However, it is a challenging task to identify a TITO system due to the interaction between loops in the system. It is considered to identify the system as having two single inputs and two single outputs, respectively. A SISO transfer function model is obtained, which becomes easier to implement.

Additionally, SISO design is said to be more comfortable to use (Nordfedlt, 2005). Being a linear square, stable, non-singular system with process industries is a property in addition to being a TITO system. This is the case when a system's input and output signals are equal in number. The system is linear and is best described as a matrix form of a linear, square, stable, non-singular transfer function. Additionally, there are attempts to linearize, to some extent, systems exhibiting non-linear behaviour.

The TITO process can generally be described as follows:

$$G_p(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \tag{1}$$

In such a case, the multiloop controller transfer function can be defined as

$$G_c(s) = \begin{pmatrix} G_{c1}(s) & 0 \\ 0 & G_{c2}(s) \end{pmatrix} \tag{2}$$

Where the transfer function of each loop can be identified as

$$G_i(s) = \frac{k_i}{a_i s^{\alpha_i} + 1} e^{\theta_i s} \tag{3}$$

Moreover, I can be referred to 1 or 2 and the variables are elements of real numbers (k_i, AI, α_i, $\theta_i \in \mathbf{R}$). In the above equation, the k_i is the static gain and AI is the time constant.

The main goal is estimating the system parameters gain, time constant, and the fractional differential orders α_I and θ_i respectively, applying measurements (data collections) of the input and output system's signals.

Considering a system with first order fractional dead time (FOFDT) model, then the TITO matrix becomes,

$$G_p(s) = \begin{bmatrix} \frac{k_{11}}{a_{11}s^{\alpha_{11}}+1}e^{\theta_{11}s} & \frac{k_{12}}{a_{12}s^{\alpha_{12}}+1}e^{\theta_{12}s} \\ \frac{k_{21}}{a_{21}s^{\alpha_{21}}+1}e^{\theta_{21}s} & \frac{k_{22}}{a_{22}s^{\alpha_{22}}+1}e^{\theta_{22}s} \end{bmatrix} \tag{4}$$

In the identification of a TITO system, it is necessary to estimate each loop separately, that is, it can be described as a SISO linear time-invariant for a fractional-order system, as given by the modeling fractional differential equation (eq. (4)). This approach, requires the system to be decoupled or presented in a diagonal form. In general, the decoupling approach compensates for the high level of interactions between the loops, which may affect the corresponding loop output signals.

For an effective system to be developed through an open loop transfer function in the TITO process, the incorporation of a decoupler is widely adopted in various methods. For an open loop transfer function to be developed between the output and the input, the corresponding loop must be closed. The other loop needs a controller allowing to develop an equivalent open loop transfer function (*eft*). This approach results in the dependence of the model on the other loop controller. The open loop transfer function model relates the output and the input, while the corresponding loop is closed through the feedback controller (say, Gc2). Assuming that the controller provides a perfect response (no time delay, time shift), then the output of the system attains a set of points with no transients (instantaneous reactions). Thus, the *eotf* for identification purposes can be written as below:

$$G_{11}^{eoft}(s) = G_{11}(s)\frac{G_{12}(s)G_{21}(s)}{G_{22}(s)} \tag{5}$$

$$G_{22}^{eoft}(s) = G_{22}(s)\frac{G_{21}(s)G_{12}(s)}{G_{11}(s)} \tag{6}$$

Hence, the equivalent model for a TITO system can be defined as

$$G_p(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \qquad (7)$$

When a decoupler exists, the TITO system is considered to behave like two independent loops, which makes the identification process simpler. However, such a technique's accuracy is highly dependent on the decoupler.

Proposed Identification Technique

To acquire only the input and output answers, this will be generated through a simple closed-loop feedback test. The obtained data can build a mathematical relationship between the input and the output response without knowing what is happening inside the system (a black box approach).

Here, the system is modelled using four independent FOFDT transfer functions. The data from the input and output responses are obtained and provided to the algorithm, which uses the integral mean square error function to give the parameters. For a cross-coupled TITO system, a relation between each transfer function and the combination of the input and output responses is used to find G_{11}, G_{12}, G_{21}, and G_{22}.

The identical structure of the TITO system has U1 and U2 inputs, and Y1 and Y2 outputs (see Fig.1). This structure consists of a closed-loop symmetrical relay that generates inputs for a system under observation. The only goal of relays is to produce continuous oscillations (known as limit cycles) thus giving ample inputs to excite the systems at desired levels. Although relays are used for identification experiments, it is not necessary to wait for convergence, as is the case with the traditional techniques for relay auto-tuning. Any step of input transient data or the first two cycles of oscillations are adequate, making identification faster without sacrificing overall efficiency.

Each transfer function of a coupled TITO model can be identified using the following combination of input and output data

$$G_p(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{Y1}{U1} & \dfrac{Y1}{U2} \\ \dfrac{Y2}{U1} & \dfrac{Y2}{U2} \end{pmatrix} \qquad (8)$$

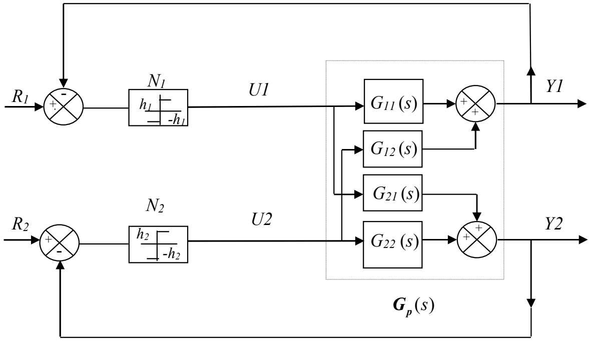

Figure 1: Relay feedback test to the active TITO system

Block Pulse Operational Matrix: The real-order linear algebraic equation can be converted into a matrix form using the block pulse operational matrix (BPOM) and for the identification problem considered here, such an operation can significantly simplify the computation (in many cases the fractional-order linear differential equations are solved by an orthogonal approach). Through block pulse functions and operational matrix, various signals can be expanded for the fractional systems (Li and Sun, 2011). When referring to block pulse functions, is a set of orthogonal functions which can be defined in the time interval [0, T] as:

$$Q_i(t) = \begin{cases} 1, & \frac{i-1}{M} \le t \le \frac{i}{M} T \\ 0, & x \ge 0 \end{cases} \tag{9}$$

where I=1,2,…,M. M is considered to be the number of elementary functions used.

Integral functions which are absolute on the time interval [0, T] can be expanded into block pulse function as such:

$$f(t) \cong f^T \psi_{\boldsymbol{M}}(t) = \sum_{i=1}^{\boldsymbol{M}} f_i \psi_{\boldsymbol{i}}(t) \tag{10}$$

From the RL definition, the block pulse basis integral functions can be obtained as follows

$$I_0^{\propto}\psi_M(t) \approx F_{\propto}\psi_M(t) \tag{11}$$

The vector which caters to block pulse basis functions with *T* transpose is described, as $\psi_M^T(t) = [\psi_1(t), \psi_2(t), ..., \psi_M(t)]$ is the generalized operator matrix for fractional integration, which as such

$$f_{\propto} = \left(\frac{T}{M}\right)^{\propto} \frac{1}{\mathrm{T}(\alpha+2)} \begin{bmatrix} f_1 & f_2 & \cdots & f_M \\ 0 & f_1 & \cdots & f_{M-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & f_1 \end{bmatrix} \tag{12}$$

This is the generalized block pulse operational matrix which enables any absolute integral function *x (t)* to be written as

$$I_0^{\propto}x(t) \approx x^T F_{\propto}\psi_M(t) \tag{13}$$

The coefficient vector is represented as $x^T = [x_1, x_2, ..., x_N]$. The main purpose of using a generalized operational matrix is that it reduces the complexity when converting the fractional integral of a function into the form of algebraic operations.

Delayed BPOM functions: Introducing a time delay $f(t-\tau)$ with an absolute integral function, will be expanded and the block pulse delay functions will be as follow

$$f(t-\tau) \cong f^T\psi_M(t-\tau) = \sum_{i=1}^{M} f_i\psi_i(t-\tau) \tag{14}$$

$f^T = [f_1, f_2, ..., f_M]$ are the coefficient vector which can be defined as:

$$f_i = \frac{M}{F}\int_0^F f_i\psi_M(t-\tau)dt \tag{15}$$

The function $\psi_M(t)$ is shifted by the delay $\psi_M(t-\tau)$, which is expressed as, $\psi_M(t-\tau) = P\psi_M(t),\ t > \tau$ and $0 \le t \le T$. Here, *P* represents the delay operational matrix for the BPF. This can be obtained by deriving the operational matrix as .

Let us take the sample time, $\frac{i-1}{M}T \le t - zl < \frac{i}{M}T$, then

$$\tau = zl = z\frac{T}{M},\ z = 1, 2, ..., M-1. \quad (16)$$

This is when delay τ is the multiplier then, $\psi_i(t-\tau) = \psi_i(t-zl)$

$$\psi_i(t-\tau) = \begin{cases} 1 & \frac{i-1}{M}T \le t - zl < \frac{i}{M}T \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

$$\psi_i(t-\tau) = \begin{cases} 1 & \frac{i-1+z}{M}T \le t < \frac{i+z}{M}T \\ 0 & \text{otherwise} \end{cases}$$

(18)

Thus, $\boldsymbol{P}_{(MXM)}$ can be represented as

$$P = \begin{pmatrix} 0 & .. & 0 & 1 & 0 & .. & .. & 0 \\ 0 & .. & 0 & 0 & 1 & .. & .. & 0 \\ : & : & : & : & : & : & \ddots & \\ 0 & 0 & 0 & 0 & 0 & .. & .. & 1 \\ : & .. & : & : & : & : & : & : \\ 0 & .. & 0 & 0 & 0 & .. & .. & 0 \end{pmatrix} \quad (19)$$

To choose an integer N, the following can be done

$$N = \begin{cases} \frac{T}{\tau} & \text{if an integer number} \\ \frac{T}{\tau} + 1 & \text{otherwise} \end{cases}$$

Then the intervals will differ

$$[0,\tau],[\tau,2\tau],....,[(N-1)\tau,N\tau] \tag{20}$$

Note that in (19) we have $N\tau \geq T$ and therefore M can be selected as

$$M = zN, \quad z = 1,2,... \tag{21}$$

Then

$$\tau = zN = z\frac{N\tau}{M} \tag{22}$$

This means that P can be computed by (19) as well as when performing Riemann-Liouville fractional integration and manipulating the equation written in a matrix form. Then, we have

$$(I_0^{\alpha}\psi_M)(t-\tau) \approx F_M\psi_M(t-\tau) = F_M P\psi_M(t) \tag{23}$$

Output from the fractional order transfer function: Now, consider the identification of a system described by a single pole fractional-order model. This needs several steps to obtain the fractional order transfer function. Now, using $G_{11}(s)$ from (4) allows one to write

$$Y(s)[a_1 s^{\alpha} + a_0] = U(s)[b_0] \tag{24}$$

With the help of the inverse Laplace transformation and the Riemann-Liouville integral definition, we get

$$Y(t)[a_1 + a_0 I^{\alpha}] = U(t)[b_0 I^{\alpha}] \tag{25}$$

Now, the integral can be substituted by F_{α} taking into account that this step generates the algebraic terms in the matrix and therefore affects the transposes of both the input and output. The transpose of $Y(t)$ is taken and this leads to

$$Y^T = U^T F_\alpha b_0 [a_1 I + F_\alpha]^{-1} \tag{26}$$

Then, concerning Y we get

$$Y = (U^T F_\alpha b_0 [a_1 I + F_\alpha]^{-1})^T \tag{27}$$

This is the initial estimate of the parameter. The estimated system output using the operational matrices can be defined as

$$\hat{Y} = (U^T \hat{F}_\alpha \hat{b}_0 [\hat{a}_1 I + \hat{F}_\alpha]^{-1})^T \tag{28}$$

To obtain the optimal system parameters, an objective function defining the error as a difference between the initial estimate and the estimated output of the system is required (see eq.29). In this context, the initial estimation data is the input and the output response of the Simulink model which is utilized as commented next.

$$(\hat{a}_{ij}, \hat{b}_{ij}, \hat{\alpha}_{ij}, \hat{k}_{ij}, \hat{\theta}_{ij}) = \min \frac{1}{N} \sum_{t=1}^{N} [Y(t) - \hat{Y}(t)]^2 \tag{29}$$

The function (29) can be solved using Matlab: the *fsolve* function was used to obtain the optimized parameters. Moreover, for providing optimizing options, *optimset* function was employed to create (a structure or modify) a parameter structure for the optimizing parameters.

Numerical verifications

Two TITO systems were studied to verify the proposed identification scheme. After activating the plant with a relay feedback test, the input and output data were exported to the matrix laboratory's workspace with a sampling time of 512. Note that we need only one cycle of data, having the same number of samples as per a block pulse function. In the case of step input, one can choose 512 lengths of data of amplitude unity. The transfer function for a TITO system applied below has been widely used in the literature.

Example 1

$$G_p(s) = \begin{bmatrix} \frac{12.9}{1+16.7s}e^{-0.1s} & \frac{-18.9}{1+21s}e^{-s} \\ \frac{6.6}{1+10.9s}e^{-0.2s} & \frac{-19.4}{1+14.4s}e^{-0.2s} \end{bmatrix} \tag{30}$$

Example 2

$$G_p(s) = \begin{bmatrix} \frac{-2}{s^2+3s+1} & \frac{3s+2}{s^2+3s+1} \\ \frac{3s+2}{s^2+s+1} & \frac{2s+1}{s^2+5s+1} \end{bmatrix} \tag{31}$$

As it follows from Example 1, the TITO-coupled system has a time delay. For this type of system, a delay matrix is essential to identify the input delay and plot the response of a single pole fractional order model. From the delay matrix and by $y(t)$ and $\hat{y}(t)$ it is possible to identify the delay (time shift) in the single pole fractional order transfer function. The delay matrix is a square matrix of 492 elements and a diagonal within the middle of it is defined by the *padarray* command; the response of a coupled TITO system is very complex and unstable.

Controlling the coupled tank requires a better-identified model to acquire a robust controller design. The single pole fractional order transfer function allows proving the design of an accurate and robust controller. Using the data set (*Y*1, *Y*2) and (*U*1, *U2*), the model was estimated using the proposed delayed block pulse technique. The optimal values were used to generate the single pole fractional order transfer function. As it is shown in Figure 2, the identified and true model responses for each transfer function (the fractional-order model related to example 1 is given by eq. (32)) are too close to each other.

$$\hat{G}_p(s) = \begin{bmatrix} \frac{0.28}{1.23s^{1.91}+1.47}e^{-0.1s} & \frac{-1.65}{-1.34s^{2.26}+1.84}e^{-s} \\ \frac{1.37}{0.93s^{1.88}+1.19}e^{-0.2s} & \frac{-4.16}{-1.30s^{1.55}+1.25}e^{-0.2s} \end{bmatrix} \tag{32}$$

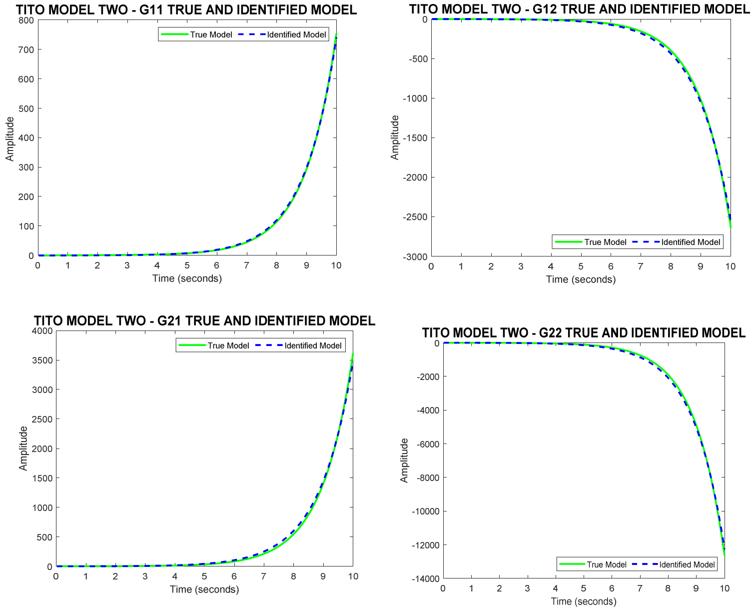

Figure 2: Estimated Model for Example 1

In the case of example 2, a non-minimum phase TITO plant is studied for verification purposes. Applying single input-output data measurements, and the proposed method, the lower order fractional transfer function for example 2 was obtained. The identified and the true model responses for each transfer function are shown in Figure 3. The fractional-order model for example 2 is given in eq. (33). The method can obtain an accurate model for the TITO-coupled plant.

$$\widehat{G}_p(s) = \begin{bmatrix} \dfrac{-10.46}{-2.59s^{1.94}+8.99}e^{-2.0s} & \dfrac{0.82}{0.72s^{1.51}+1.78}e^{-s} \\ \dfrac{1.49}{0.51s^{1.38}+1.19}e^{-0.1s} & \dfrac{-2.02}{-1.07s^{1.90}+2.63}e^{-0.2s} \end{bmatrix} \qquad (33)$$

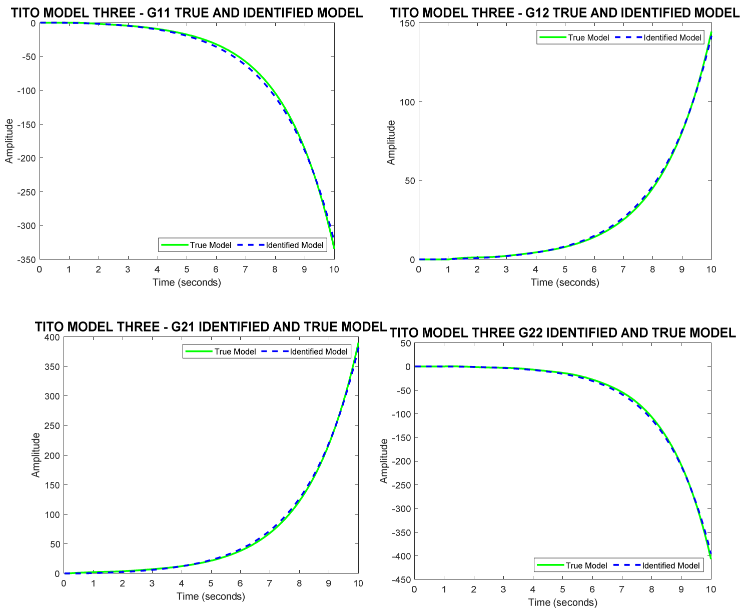

Figure 10.3: Estimated Model for Example 2

Conclusions

The R-L integral derivative was successfully used with the block pulse operational matrix to identify the TITO system. Using the delays matrix, the delay in the system was also identified with the delay matrix in the numerator of the identified response. Identification can be done simply through the input and output data of the complex system. Using the delays matrix, the delay in the system was also identified with the delay matrix in the numerator of the identified response. This method may be very useful for identifying a much more complex system with multiple loops or a complex physical structure. Identification can be done simply through the input and output data of the complex system. It is worth exploring in the future the verification of a technique with corrupted output data.

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