Square Roots and Powers in Constructive Banach Algebra Theory

Douglas S. Bridges[1] and Robin S. Havea[2]

[1] Department of Mathematics & Statistics, University of Canterbury, Private Bag 4800, Christchurch, New Zealand
d.bridges@math.canterbury.ac.nz
[2] Department of Mathematics & Computing Science, University of the South Pacific, Suva, Fiji
robin.havea@usp.ac.fj

Abstract. Several new and improved results about positive integral powers of hermitian elements, and square roots of positive elements, in a Banach algebra are proved constructively.

1 Introduction

The purpose of this article is to extend our earlier constructive[1] work on hermitian and positive elements of a separable complex Banach algebra B with identity e [6,5,9]. In particular, we provide conditions—one of which was, unfortunately, lacking in Theorem 4.2 of [6] and the corresponding result in [9]—under which we can prove constructively that positive integral powers of a hermitian element are hermitian; also, we substantially generalise, by a relatively elementary proof, the result in [5] that yields the existence and uniqueness of the square root of a positive element of B.

Although we shall refer to the [6] for much of the background material needed for this paper, for the reader's convenience we here re-present some important notions. First, let B' denote the dual of B. In general, we cannot prove that every $f \in B$ is **normed** in the sense that $\|f\| \equiv \sup\{|f(x)| : x \in B, \|x\| \leqslant 1\}$ exists. However, even when f need not be normed, we adopt the shorthand $\|f\| \leqslant c$ to signify that $|f(x)| \leqslant c$ for all $x \in B$ with $\|x\| \leqslant 1$. An element f of B' is **nonzero** if $|f(x)| > 0$ for some $x \in B$. For each dense sequence $(x_n)_{n\geqslant 1}$ in B we introduce the corresponding **double norm** on B', defined by $|||f||| \equiv \sum_{n=1}^{\infty} 2^{-n} |f(x_n)|$. The topology induced by this norm on the unit ball $B_1' \equiv \{f \in B' : \|f\| \leqslant 1\}$ of B' is independent of the dense sequence relative to which the double norm is defined, and is, in fact, the weak* topology on B_1'.

Now, we may not be able to prove constructively that the **state space** $V_B = \{f \in B' : f(e) = 1 = \|f\|\}$ of B is inhabited (that is, contains an element), let alone weak* compact as it is classically. For this reason we introduce, for each $t > 0$, the approximation

[1] We work entirely within the framework of Bishop-style constructive analysis (**BISH**—for more on which, see [2,7,8]).

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$$V_B^t = \{f \in B' : \|f\| \leqslant 1, \ |1 - f(e)| \leqslant t\}$$

to V_B. The constructive Hahn-Banach theorem ([8], Theorem 5.3.3) is strong enough for us to prove that V_B^t is inhabited; moreover, it is weak* compact for all but countably many $t > 0$. We say that $t > 0$ is **admissible** if V_B^t is weak* compact. We describe V_B as **firm** if (i) it is weak*compact and (ii) for each $\varepsilon > 0$, there exists an admissible $t > 0$ such that for each $f \in V_B^t$, there exists $g \in V_B$ with $|||f - g||| < \varepsilon$. The following result is proved in [6] (Proposition 2.4).

Proposition 1. *If B has firm state space, then so does each Banach subalgebra of B.*

We call an element x of B **hermitian** if for each $\varepsilon > 0$, there exists $t > 0$ such that $|\mathrm{Im}\, f(x)| < \varepsilon$ for all $f \in V_B^t$; and **positive**—when we write $x \geqslant 0$—if for each $\varepsilon > 0$, there exists $t > 0$ such that $\mathrm{Re}\, f(x) \geqslant -\varepsilon$ and $|\mathrm{Im}\, f(x)| < \varepsilon$ for all $f \in V_B^t$. These definitions of *hermitian* and *positive* are classically equivalent to the standard classical ones found in [3], which are constructively too weak to be of much use. The following appears as Lemma 4.1 of [6].

Lemma 1. *Suppose that V_B is firm. Then $a \in B$ is hermitian if and only if $f(a) \in \mathbf{R}$ for each $f \in V_B$, and $a \geqslant 0$ if and only if $f(a) \geqslant 0$ for each $f \in V_B$.*

By a **character** of B we mean a nonzero multiplicative linear functional $u : B \to \mathbf{C}$; such a mapping satisfies $u(e) = 1$ and is normed, with $\|u\| = 1$. The **character space** Σ_B of B comprises all characters of B and is a subset of the unit ball of B'. We cannot prove constructively that the character space of every commutative Banach algebra B is inhabited, let alone that, as classically, it is weak* compact; see page 452 of [2]. However, as we shall see in Proposition 4, we can construct elements of Σ_B under certain conditions on B.

We say that an element x of B is **strongly hermitian** (resp. **strongly positive**) if it is hermitian (resp., positive) and the state space of the closed subalgebra A of B generated by $\{e, x\}$ is the closed convex hull of Σ_A. Classically, the latter condition always holds (see [3], page 206, Lemma 3), so every hermitian (resp. positive) x is strongly hermitian (resp. strongly positive). The main results of this paper are the following.

Theorem 1. *Let B be have firm state space, and let a be a strongly hermitian element of B. Then a^n is hermitian, and a^{2n} is positive, for each positive integer n.*

Theorem 2. *Let B be have firm state space. Let a be a strongly positive element of B, and A the Banach algebra generated by $\{e, a\}$. Then there exists a unique positive element x of A such that $x^2 = a$.*

The first of these is a corrected version of [6] (Theorem 4.2), in which we should have had a hypothesis ensuring that the product of two positive elements of

A is positive. Theorem 2 replaces the restrictive requirement that B be semisimple, used in [5] (Theorem 3), by the more widely applicable strong positivity hypothesis on a.

2 Products of Hermitian/Positive Elements

When is the product of two hermitian/positive elements of a Banach algebra hermitian/positive?

Proposition 2. *Let B be commutative, with firm state space, and suppose that V_B is the weak*-closed convex hull of Σ_B. Then the product of two hermitian elements of B is hermitian, and the product of two positive elements is positive.*

Proof. Let x and y be hermitian elements of B. Given $f \in V_B$ and $\varepsilon > 0$, pick elements u_k $(1 \leqslant k \leqslant m)$ of Σ_B, and corresponding nonnegative numbers λ_k, such that $\sum_{k=1}^{m} \lambda_k = 1$ and $|f(xy) - \sum_{k=1}^{m} \lambda_k u_k(xy)| < \varepsilon$. Since $\Sigma_B \subset V_B$, we have $u_k(x), u_k(y) \in \mathbf{R}$, by Lemma 1; whence

$$|\mathrm{Im}\, f(xy)| \leqslant \left| \sum_{k=1}^{m} \lambda_k \,\mathrm{Im}\, u_k(xy) \right| + \left| f(xy) - \sum_{k=1}^{m} \lambda_k u_k(xy) \right| < 0 + \varepsilon = \varepsilon.$$

But $\varepsilon > 0$ is arbitrary, so $\mathrm{Im}\, f(xy) = 0$ and therefore $f(xy) \in \mathbf{R}$. Moreover, if $x \geqslant 0$ and $y \geqslant 0$, then

$$\mathrm{Re}\, f(xy) \geqslant \sum_{k=1}^{m} \lambda_k \,\mathrm{Re}\, (u_k(x) u_k(y)) - \left| f(xy) - \sum_{k=1}^{m} \lambda_k u_k(xy) \right| > 0 - \varepsilon = -\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows from Lemma 1 that, when x, y are hermitian, $f(xy) \in \mathbf{R}$, and when they are positive, $f(xy) \geqslant 0$.

We are now prepared for the **Proof of Theorem 1.** Under its hypotheses, let A be the closed subalgebra of B generated by $\{e, a\}$. By Proposition 1, V_A is firm; since a is strongly hermitian, the hypotheses of Proposition 2 are satisfied, and the desired conclusions follow almost immediately. ■

By a **positive linear functional** on B we mean an element f of B' such that $f(x) \geqslant 0$ for each positive element of B; we write $f \geqslant 0$ to signify that f is positive. Every element of the state space V_B is positive (see Section 3 of [6]).

Consider a convex subset K of B'_1. We say that $f \in K$ is a **classical extreme point** of K if

$$\forall_{g,h \in K} \left(f = \tfrac{1}{2}(g+h) \Rightarrow g = h = f \right),$$

and an **extreme point** of K if

$$\forall_{\varepsilon > 0}\, \exists_{\delta > 0} \forall_{g,h \in K} \left(\left|\left|\left| f - \tfrac{1}{2}(g+h) \right|\right|\right| < \delta \Rightarrow |||g - h||| < \varepsilon \right).$$

An extreme point is a classical extreme point, and the converse holds classically if K is also weak* compact. If f is an extreme point of K relative to one double norm on B', then it is an extreme point relative to any other double norm on B'. Proposition 3.1 of [6] states that if the state space V_B is firm, then every extreme point of V_B is an extreme point of the convex set $K^0 = \{f \in B' : f \geqslant 0, \, f(e) \leqslant 1\}$.

We omit the proof of our next result, which is very close to that on page 38 of [10].

Lemma 2. *Suppose that B is commutative and that the product of two positive elements of B is positive. Let f be a classical extreme point of K^0. Then $f(xy) = f(x)f(y)$ for all $x \in B$ and all positive $y \in B$.*

Proposition 3. *Suppose that B is generated by commuting positive elements, and that the product of two positive elements of B is positive. Then every classical extreme point of K^0 is a multiplicative linear functional on B.*

Proof. Clearly, B is commutative. Let f be a classical extreme point of K^0, and consider first the case where f is nonzero. A simple induction based on Lemma 2 shows that

$$f(xy^n) = f(x)f(y)^n \qquad (x \in B, y \in B, y \geqslant 0)\,. \tag{1}$$

Given any $x, y \in B$ and $\varepsilon > 0$, pick positive elements $a_1, \ldots, a_n$ and a complex polynomial $p(\zeta_1, \ldots, \zeta_n)$ such that $\|y - z\| < \varepsilon$, where $z \equiv p(a_1, \ldots, a_n)$. Since finite products of positive elements of B are positive, we see from (1) that $f(xz) = f(x)f(z)$; whence

$$\begin{aligned} |f(xy) - f(x)f(y)| &\leqslant |f(xy - xz)| + |f(x)f(z) - f(x)f(y)| \\ &\leqslant \|x\,(y - z)\| + |f(x)|\,|f(z - y)| \\ &\leqslant 2\,\|x\|\,\|y - z\| \leqslant 2\,\|x\|\,\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f(xy) = f(x)f(y)$. Finally, to remove the condition that f be nonzero, let $x, y \in B$ and suppose that $f(xy) \neq f(x)f(y)$. Then, by the foregoing, f cannot be nonzero; so $f = 0$ and therefore $f(xy) = 0 = f(x)f(y)$, a contradiction. Thus we have $\neg\,(f(xy) \neq f(x)f(y))$ and therefore $f(xy) = f(x)f(y)$.

Proposition 4. *Suppose that B is generated by commuting positive elements and has firm state space, and that the product of two positive elements of B is positive. Let A be a unital Banach subalgebra of B. Then Σ_A is inhabited, and V_A is the double-norm-closed convex hull of Σ_A.*

Proof. By Proposition 1, V_A is firm and hence compact. Since V_A is also convex, it follows from the Krein-Milman theorem ([2], page 363, Theorem (7.5)) that it has extreme points and is the double-norm-closed convex hull of the set of those extreme points. By Proposition 3.1 of [6], every extreme point of V_A is an extreme point, and hence a classical extreme point, of K^0. Since the elements of V_A are nonzero, the result now follows from Proposition 3.

Proposition 4 readily yields a partial converse of Proposition 2:

Corollary 1. *Let a be an element of B all of whose positive integer powers are positive, and let A be the closed subalgebra of B generated by $\{e, a\}$. Then Σ_A is inhabited, and V_A is the double-norm-closed convex hull of Σ_A.*

When—as in the Banach algebra $C(X)$, where X is a compact metric space—is a hermitian element expressible as a difference of positive elements? To answer this, we need to say more about approximations to the character space.

For any dense sequence $(x_n)_{n\geqslant 1}$ in B, we can find a strictly decreasing sequence $(t_n)_{n\geqslant 1}$ of positive numbers converging to 0 such that for each n the set

$$\Sigma_B^{t_n} \equiv \{u \in B_1' : |u(x_j x_k) - u(x_j)u(x_k)| \leqslant t_n \ \ (1 \leqslant j, k \leqslant n) \\ \wedge \ |1 - u(e)| \leqslant t_n\}$$

is (inhabited and) weak* compact ([2], page 460, Proposition (2.7)). The intersection of these sets is the character space Σ_B. For each $x \in B$ we define

$$\|x\|_{\Sigma_B^{t_n}} \equiv \sup\left\{|u(x)| : u \in \Sigma_b^{t_n}\right\},$$

which exists since the function $x \rightsquigarrow |u(x)|$ is uniformly continuous on the double-norm compact set $\Sigma_b^{t_n}$.

We recall two important result from constructive Banach algebra theory.

Proposition 5. Sinclair's theorem: *If x is a hermitian element of the Banach algebra B, then $\|x^n\|^{1/n} = \|x\|$ for each positive integer n* ([4], pages 293–303).

Proposition 6. *Let B be commutative, and let $(t_n)_{n\geqslant 1}$ be a decreasing sequence of positive numbers converging to 0 such that Σ^{t_n} is compact for each n. Then the sequences $\left(\|x^n\|^{1/n}\right)_{n\geqslant 1}$ and $(\|x\|_{\Sigma^{t_n}})_{n\geqslant 1}$ are **equiconvergent:** that is, for each term a_m of one sequence and each $\varepsilon > 0$, there exists N such that $b_n \leqslant a_m + \varepsilon$ whenever b_n is a term of the other sequence with $n \geqslant N$* ([2], Chapter 9, Proposition (2.9)).

Of particular importance for us is the following:

Corollary 2. *Let B be commutative, and let $(t_n)_{n\geqslant 1}$ be a decreasing sequence of positive numbers converging to 0 such that $\Sigma_B^{t_n}$ is compact for each n. Let h be a hermitian element of B. Then $\lim_{n\to\infty} \|h\|_{\Sigma_B^{t_n}} = \|h\|$.*

Proof. By Sinclair's theorem, $\|h^n\|^{1/n} = \|h\|$ for each positive integer n. It follows from Proposition 6 that for each $\varepsilon > 0$, there exists N such that $\|h\|_{\Sigma_B^{t_n}} < \|h\| + \varepsilon$ for all $n \geqslant N$. By that same proposition, for each $n \geqslant N$, there exists m such that $\|h\| = \|h^m\|^{1/m} \leqslant \|h\|_{\Sigma_B^{t_n}} + \varepsilon$. Hence $\left|\|h\| - \|h\|_{\Sigma_B^{t_n}}\right| < \varepsilon$ for all $n \geqslant N$.

Lemma 3. *If x, y are commuting hermitian elements of B, then*

$$\max\{\|x\|, \|y\|\} \leqslant \|x+iy\|.$$

Proof. Replacing B by the closed subalgebra generated by $\{e, x, y\}$, let $(t_n)_{n\geqslant 1}$ be a decreasing sequence of positive numbers converging to 0 such that Σ^{t_n} is compact for each n. Since $\Sigma_B^{t_n} \subset V_B^{t_n}$ and x, y are hermitian, there exists N such that $\min\{|\operatorname{Im} u(x)|, |\operatorname{Im} u(y)|\} < \varepsilon$ for each $n \geqslant N$ and each $u \in \Sigma_B^{t_n}$. For such u we have

$$|u(x+iy)| \geqslant |\operatorname{Re} u(x+iy)| = |\operatorname{Re} u(x) - \operatorname{Im} u(y)| > |\operatorname{Re} u(x)| - \varepsilon$$

and therefore $|\operatorname{Re} u(x)| < |u(x+iy)| + \varepsilon \leqslant \|x+iy\| + \varepsilon$. But

$$|u(x)|^2 = (\operatorname{Re} u(x))^2 + (\operatorname{Im} u(x))^2 < |\operatorname{Re} u(x)|^2 + \varepsilon^2,$$

so $|u(x)|^2 \leqslant (\|x+iy\| + \varepsilon)^2 + \varepsilon^2$. Since $u \in \sum_B^{t_n}$ is arbitrary, we conclude that $\|x\|_{\Sigma_B^{t_n}}^2 \leqslant (\|x+iy\| + \varepsilon)^2 + \varepsilon^2$. Now, x is hermitian, so by Sinclair's theorem, $\|x^n\|^{1/n} = \|x\|$ for each positive integer n. It follows from Corollary 2 that $\|x\|^2 = \lim_{n\to\infty} \|x\|_{\Sigma_B^{t_n}}^2 \leqslant \|x+iy\|^2$, and hence that $\|x\| \leqslant \|x+iy\|$. Finally, replacing x, y by $-y, x$ in the foregoing, we obtain $\|y\| \leqslant \|-y+ix\| = \|x+iy\|$.

Proposition 7. *Suppose that B is generated by commuting positive elements, and that the product of two positive elements of B is positive. Then for each hermitian element x of B and each $\varepsilon > 0$, there exist positive $a, b \in B$ with $\|x-(a-b)\| < \varepsilon$.*

Proof. There exist commuting positive elements $z_1, \ldots, z_m$ of B with each $\|z_k\| \leqslant 1$, and a polynomial $p(\zeta_1, \ldots, \zeta_m)$ over $\mathbf{C}$, such that

$$\|x - p(z_1, \ldots, z_m)\| < \varepsilon.$$

Write

$$p(\zeta_1, \ldots, \zeta_m) \equiv \sum_{i_1,\ldots,i_m=1}^{n} \alpha(i_1, \ldots, i_m)\zeta_1^{i_1} \cdots \zeta_m^{i_m}$$

where each $\alpha(i_1, \ldots, i_m) \in \mathbf{C}$. Note that each term $z_1^{i_1} \cdots z_m^{i_m}$ is positive. Perturbing each coefficient $\alpha(i_1, \ldots, i_m)$ by a sufficiently small amount, we can arrange that Re $\alpha(i_1, \ldots, i_m) \neq 0$ and Im $\alpha(i_1, \ldots, i_m) \neq 0$ for each tuple $(i_1, \ldots, i_m)$. Let

$$\begin{aligned}
P &\equiv \{(i_1, \ldots, i_m) : \operatorname{Re} \alpha(i_1, \ldots, i_m) > 0\}, \\
Q &\equiv \{(i_1, \ldots, i_m) : \operatorname{Re} \alpha(i_1, \ldots, i_m) < 0\}, \\
a &\equiv \sum_{(i_1,\ldots,i_m)\in P} \operatorname{Re} \alpha(i_1, \ldots, i_m) z_1^{i_1} \cdots z_m^{i_m}, \\
b &\equiv - \sum_{(i_1,\ldots,i_m)\in Q} \operatorname{Re} \alpha(i_1, \ldots, i_m) z_1^{i_1} \cdots z_m^{i_m}.
\end{aligned}$$

Then $a \geqslant 0$ and $b \geqslant 0$. Moreover,

$$\begin{aligned}\varepsilon &> \|x - p(z_1, \ldots, z_m)\| \\ &= \left\| x - (a-b) - i \left(\sum_{i_1,\ldots,i_m=1}^{n} (\operatorname{Im} \alpha(i_1, \ldots, i_m))\, z_1^{i_1} \cdots z_m^{i_m} \right) \right\|\end{aligned}$$

where both $x - (a-b)$ and $\sum_{i_1,\ldots,i_m=1}^{n} (\operatorname{Im} \alpha(i_1, \ldots, i_m))\, z_1^{i_1} \cdots z_m^{i_m}$ are hermitian; whence $\|x - (a-b)\| < \varepsilon$, by the preceding lemma.

The approximation of hermitian elements by differences of two positive elements, as in the preceding proposition, is related to classical work on V-algebras and the Vidav-Palmer theorem (see, in particular, Lemma 8 in §38 of [3]). We intend exploring that further in a future paper.

3 The Path to Theorem 2

Our proof of Theorem 2 requires yet more preliminaries, beginning with an estimate that will lead to the continuity of positive square root extraction.

Lemma 4. *Let p be a positive element of the Banach algebra B such that $\|p\| \leqslant 1$, and let A be the Banach algebra generated by $\{e, p\}$. Let $0 < \delta_1, \delta_2 \leqslant 1$, and suppose that there exist positive elements b_1, b_2 of A such that $b_1^2 = e - \delta_1 p$ and $b_2^2 = e - \delta_2 p$. Then*

$$\|b_1 - b_2\|^2 \leqslant \frac{68}{3} |\delta_1 - \delta_2| (1 + \|p\|).$$

Proof. Given $\varepsilon > 0$, let

$$\alpha = \sqrt{\frac{1}{3} (|\delta_1 - \delta_2| (1 + \|p\|) + 2\varepsilon^2)}.$$

Pick $t_0 > 0$ such that: $V_A^{t_0}$ and $\Sigma_A^{t_0}$ are compact,

$$\left|u(b_1^2) - u(b_1)^2\right| < \alpha^2 \text{ and } \left|u(b_2^2) - u(b_2)^2\right| < \alpha^2 \text{ for each } u \in \Sigma_A^{t_0}, \text{ and}$$

$$\min\{\operatorname{Re} f(b_1), \operatorname{Re} f(b_2)\} \geqslant -\alpha \ \text{ and } \ \max\{\operatorname{Im} f(b_1), f(b_2)\} \leqslant \alpha \text{ for each } f \in V_A^{t_0}$$

For each $u \in \Sigma_A^{t_0}$ we have

$$\begin{aligned}&|u(b_1 - b_2)|\, |u(b_1 + b_2)| = \left|u(b_1)^2 - u(b_2)^2\right| \\ &\leqslant \left|u\left(b_1^2 - b_2^2\right)\right| + \left|u(b_1^2) - u(b_1)^2\right| + \left|u(b_2^2) - u(b_2)^2\right| \\ &< \left\|b_1^2 - b_2^2\right\| + 2\varepsilon^2 = |\delta_1 - \delta_2| (1 + \|p\|) + 2\varepsilon^2 = 3\alpha^2.\end{aligned}$$

Either $|u(b_1 - b_2)| < 2\alpha$ or $|u(b_1 - b_2)| > \alpha$. In the latter case,

$$|\operatorname{Re} u(b_1) + \operatorname{Re} u(b_2)| \leqslant |u(b_1 + b_2)| < 3\alpha.$$

Suppose that $\operatorname{Re} u(b_1) > 4\alpha$. Then $\operatorname{Re} u(b_2) < 3\alpha - \operatorname{Re} u(b_1) < -\alpha$, which, since $u \in V_A^{t_0}$, contradicts our choice of t_0. Hence $-\alpha \leqslant \operatorname{Re} u(b_1) \leqslant 4\alpha$ and therefore $|\operatorname{Re} u(b_1)| \leqslant 4\alpha$; so

$$\left|u(b_1)^2\right| = (\operatorname{Re} u(b_1))^2 + (\operatorname{Im} u(b_1))^2 \leqslant 17\alpha^2$$

and therefore $|u(b_1)| \leqslant \sqrt{17}\alpha$. Likewise, $|u(b_2)| \leqslant \sqrt{17}\alpha$, so $|u(b_1 - b_2)| \leqslant 2\sqrt{17}\alpha$, an inequality that also holds in the case $|u(b_1 - b_2)| < 2\alpha$. Since $u \in \Sigma_A^{t_0}$ is arbitrary, we now see that $\|b_1 - b_2\|_{\Sigma_A^{t_0}}^2 < 68\alpha^2$. But $b_1 - b_2$ is hermitian, so, by Corollary 2,

$$\|b_1 - b_2\|^2 \leqslant \|b_1 - b_2\|_{\Sigma_A^{t_0}}^2 < \frac{68}{3}\left(|\delta_1 - \delta_2|(1 + \|p\|) + 2\varepsilon^2\right).$$

Since $\varepsilon > 0$ is arbitrary, we now obtain the desired conclusion.

Proposition 8. *Let B have firm state space, let a be a strongly positive element of B such that $\|a\| < 1$, and let A be the Banach algebra generated by $\{e, a\}$. Then there exists a positive element s of A such that $s^2 = e - a$.*

Proof. Our proof is based on that of Bonsall and Duncan [3] (page 207, Lemma 7). Those authors use the Gelfand representation theorem and Dini's theorem, the latter lying outside the reach of **BISH** (see [1]). However, we can avoid those two theorems altogether, as follows. First, we note that, by Lemma 5 of [6], $e - a \geqslant 0$ and $\|e - a\| \leqslant 1$. Consider the special case where $\|a\| < 1$. Let $x_0 = 0$ and, for each n,

$$x_{n+1} = \frac{1}{2}(a + x_n^2). \tag{2}$$

A simple induction shows that x_n belongs to A. Noting that $x_1 = \frac{1}{2}a$, suppose that $\|x_n\| < \|a\|$; then

$$\|x_{n+1}\| \leqslant \frac{1}{2}\left(\|a\| + \|x_n\|^2\right) < \frac{1}{2}\left(\|a\| + \|a\|^2\right) = \frac{1 + \|a\|}{2}\|a\| < \|a\|.$$

Thus $\|x_n\| < \|a\|$ for each n. Next, observe that, by Proposition 1, V_A is firm; it follows from Proposition 2 that the product of two positive elements of A is positive. Thus if $x_n \geqslant 0$, then $x_n^2 \geqslant 0$, so $a + x_n^2 \geqslant 0$ and therefore $x_{n+1} \geqslant 0$; since $x_0 \geqslant 0$, we conclude that $x_n \geqslant 0$ for each n. In particular, $x_1 - x_0 = x_1 \geqslant 0$. Now suppose that $x_n - x_{n-1} \geqslant 0$. Then since x_n, x_{n-1},and therefore $x_n + x_{n-1}$ are all positive elements of A,

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n^2 - x_{n-1}^2\right) = \frac{1}{2}(x_n + x_{n-1})(x_n - x_{n-1}) \geqslant 0.$$

Moreover,

$$\|x_{n+1} - x_n\| \leqslant \frac{1}{2}(\|x_n\| + \|x_{n-1}\|)\|x_n - x_{n-1}\| \leqslant \|a\|\|x_n - x_{n-1}\|,$$

so, by another induction, $\|x_{n+1} - x_n\| \leqslant \|a\|^n \|x_1\| = \frac{1}{2}\|a\|^{n+1}$. It follows that if $m > n \geqslant 1$, then

$$\|x_m - x_n\| \leqslant \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \leqslant \sum_{k=n}^{m-1} \frac{1}{2}\|a\|^{k+1}$$
$$\leqslant \frac{1}{2}\|a\|^{n+1} \sum_{k=0}^{\infty} \|a\|^k = \frac{\|a\|^{n+1}}{2(1-\|a\|)} \to 0 \text{ as } n \to \infty.$$

Hence $(x_n)_{n \geqslant 1}$ is a Cauchy sequence in the Banach algebra A and therefore converges to a limit $x \in A$. Clearly, x is positive, $\|x\| \leqslant \|a\| < 1$, and x commutes with a. Letting $n \to \infty$ in (2), we obtain $x = \frac{1}{2}(a + x^2)$. Hence $(e-x)^2 = e - 2x + (2x - a) = e - a$. Moreover, $e - x \in A$ and, by Lemma 5 of [6], $e - x \geqslant 0$. Thus $s \equiv e - x$ is a positive square root of $e - a$ in A.

Now consider the general case where $\|a\| \leqslant 1$. For each integer $n \geqslant 2$ set $r_n = 1 - n^{-1}$. Then $r_n a \geqslant 0$ and $\|r_n a\| < 1$. By the foregoing, there exists a positive element s_n of A with $s_n^2 = e - r_n a$. Taking $p = e - a, \delta_1 = \frac{1}{m}$, and $\delta_2 = \frac{1}{n}$ in Lemma 4 now yields

$$\|s_m - s_n\|^2 \leqslant 68 \left| \frac{1}{m} - \frac{1}{n} \right| (1 + \|a\|),$$

from which we see that $(s_n)_{n \geqslant 1}$ is a Cauchy sequence in A. Since A is complete, this sequence has a limit $s \in A$. Clearly, $s \geqslant 0$ and $s^2 = e - a$. Finally, taking $p = e - a$ and $\delta_1 = \delta_2 = 1$ in Lemma 3, we see that s is the unique positive square root of $e - a$ in A.

Finally, we have the **Proof of Theorem 2.** Under the hypotheses of that theorem, if $\|a\| \leqslant 1$, then by Lemma 5 of [5], $e - a \geqslant 0$ and $\|e - a\| \leqslant 1$; whence, by Proposition 8, there exists a unique positive element b of A such that $b^2 = e - (e - a) = a$. In the general case, compute $\delta > 0$ such that $\|\delta a\| \leqslant 1$. There exists a unique positive element p of A such that $p^2 = \delta a$. Then $\delta^{-1/2} p$ is a positive element of A, and $(\delta^{-1/2} p)^2 = a$. Moreover, if b is a positive square root of a, then $\delta^{1/2} b$ is a positive square root of δa, so $\delta^{1/2} b = p$ and therefore $b = \delta^{-1/2} p$. This establishes the uniqueness of the positive square root of a. ■

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