A Lagrangian-based Swarming Behavior in the Absence of Obstacles

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July 20, 2010

Abstract

Lagrangian swarm models consider long-range attraction and short-range repulsion between individuals, moving with the velocity of the swarm's centroid, as a seed in the formation of the swarm itself and its behavior. By constructing a Lyapunov function based on this heuristic rule, we create a relatively simple gradient system which surprisingly exhibits complex emergent or self-organized motions in the absence of fixed or moving obstacles. The Lyapunov function contains an inter-individual collision-avoidance component; hence the component is bounded, yet it guarantees collision avoidance. Three parameters are utilized, and which we call *cohesion parameter*, *coupling parameter*, and *convergence parameter*. They are, respectively, a measure of the strengths of the cohesion of the swarm, the interaction between any two individuals and the instantaneous velocity of an individual with respect to the swarm centroid. By varying these parameters in a precise way, computer simulations show that for a sufficiently large number of individuals, our proposed model generates four types of swarming-like behaviors. They are (1) the cruise formation (linear or nonlinear) reminiscent of a cruising and leaderless school of fish, or a moving herd of land animals with a leader (leader-following), (2) random walks similar to the swarming behavior of fruit flies *Drosophila melanogaster*, (3) constant arrangements requiring individuals to aggregate and stop, as in fruiting body formation by bacteria, and (4) circular motions reminiscent of the behavior of a school of fish when threatened by a predator.

1 Introduction

With the amount of work carried out over the last three decades on studying and modelling swarms, beginning with the work of Okubo (1980), it is now possible to group different modelling approaches into at least two; the *Eulerian* and the *Lagrangian* approaches. In the Eulerian approach, the swarm is considered a *continuum* described by its density. In the Lagrangian approach, the state (position, instantaneous velocity and instantaneous acceleration) of each individual and its relationship with other individuals in the swarm is studied; it is an *individual-based* modelling. A question, posed by Edelstein-Keshet (2001) in her descriptive survey of mathematical models of swarming and social aggregation, vividly elucidates the dichotomy between the two; "*are we following a given individual to see how it is affected by its neighbors, or are we watching the herd move past us as a density wave*?" Edelstein-Keshnet and colleagues indeed provided two separate illuminating papers on a continuum model and a Lagrangian model for swarms [Mogilner and Edelstein-Keshet (1999); Mogilner et al (2003)]. Excellent reviews of these approaches and their advantages and disadvantages can be found in Gazi and Passino (2004b) and Merrifield (2006).

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Our interest lies in utilizing the key component of the Lagrangian approach, and that is, the use of attraction and repulsion functions to model the swarming behavior in which there is a *long-range attraction* and a *short-range repulsion* between individuals in the swarm [Edelstein-Keshet (2001), Gazi and Passino (2004b)]. This behavior leads to aggregation and formation, which are important for the survival of the members of the swarm [Brown and Warburton (1999); Ekanayaka and Pathirana (2009)]. If constructed appropriately, these attraction and repulsive functions, can be expressed as a gradient of some *artificial or social potential function*. This means that there is a Lyapunov function, a minimum of which corresponds to a stable stationary state of the individual-based Lagrangian system [Edelstein-Keshet (2001)]. As noted in Gazi and Passino (2003, 2004b), the use of a gradient system ensures there is an element of distribution of tasks among the members of the swarm and that the swarm members are performing *distributed optimization*. Indeed, because of the existence of the Lyapunov function, each individual in the swarm is individually and optimally searching a minimum. The stability conditions provided by the Lyapunov function can also provide the *cohesiveness* of the swarm in which the distances between individuals in the swarm are bounded from above [Mogilner et al (2003)].

The model by Mogilner et al (2003) uses a class of attraction and repulsion functions that are formed using both exponential and power laws. A recent stream of modelling that utilizes the same basic form of this class of interaction functions is traceable to the work of Gazi and Passino (2003). The attraction-repulsion function has an attraction component that dominates for large distances and a repulsion component that dominates for small distances. The stability conditions, provided by a Lyapunov function, are used to estimate the size of the swarms. In 2004, the authors extended their 2003 results by also considering interactions between individuals and their environment [Gazi and Passino (2004b)]. Specifically, they considered a swarm that is moving in a profile of nutrients or toxic substances, i.e., an attractant/repellent profile. Also in 2004, the authors provided another type of attraction-repulsion function [Gazi and Passino (2004a)].

The 2003 Gazi-Passino model is *isotropic*; there is uniformity in attraction or repulsion between all members of the swarm. Moreover, it is *reciprocal*; every member i moves toward every other member j exactly the same amount as j moves toward i. In 2003, Chu and colleagues generalized the Gazi-Passino model to include anisotropy [Chu et al (2003)]. Their *anisotropic* model contains a *coupling matrix* that allows the interaction strength between individuals in a swarm to vary. They assumed that the interactions between *only* at least two individuals, and not all, were reciprocal. In 2004, Wang and colleagues removed this reciprocity argument by adopting an asymmetric coupling matrix [Wang et al (2004)].

A shortcoming of the the Gazi-Passino model and its variants mentioned above is that they do not have a collision-avoidance capability between members of the swarm because the attraction-repulsion functions does not grow to infinity when individuals collide. The effect of their attraction-repulsion functions is only enough for each individual to move towards the center of the swarm and stop without collapsing to a tight cluster. To resolve this issue, Liu and colleagues, in 2005, introduced a repulsion term which is inversely proportional to the forth-power of the distance between two individuals [Liu et al (2005)]. In 2009, they expanded their work to obtain swarm models that are non-reciprocal and exhibit self-organized oscillations [Liu et al (2009)].

In 2006, Chen and Fang (2006a,b) added a component to a Geza-Passino-like model, and produced a system that is practically *scalable*, in the sense that regardless of how large the size of the swarm is, there is no or limited cost associated with any increase in size. This is in contrast to other Geza-Passino-like models in which every member knows the state of every other member in the swarm. However, Gazi and Passino (2004b) has argued that sensing limitations in engineering applications, like controlling robots, could be solved with technologies such as the Global Positioning System. Indeed, as reported in Martinoli et al (2004), distributed control principles had been successfully applied to a series of case studies in collective robotics (aggregation and segregation, foraging, collaborative stick pulling, cooperative transportation, flocking and navigation in formation, odor source localization, cooperative mapping, and soccer tournaments) for which several approaches extensively exploited global communication capabilities.

Another stream of modelling within the Lagrangian framework uses algebraic graph theory, potential functions and the Lyapunov method to study flocking. The work of Olfati-Saber (2006) provides distributed and decentralized algorithms with obstacle avoidance capabilities. In 2007, Tanner and colleagues used, in addition, non-smooth analysis to construct discontinuous controllers that ensure a robust flock model [Tanner et al (2007)]. In 2008, Gu and Hu used fuzzy logic and the algebraic graph theory, together with non-smooth analysis, to create functions for collision avoidance between members in a flock [Gu and Hu (2008)]. These three scalable models express the three well-known heuristic flocking rules of Reynolds (1987) into precise mathematical statements.

Our approach is also Lagrangian; hence, we consider spacing between individuals, which moves with the velocity of the swarm centroid, of primary importance. We create functions to measure the distance between individuals, and use them to move the individuals toward the swarm centroid and for collision-avoidance between the members of the swarm, and for avoidance of any fixed obstacle in the swarm's path. We do this by having these functions as part of a Lyapunov function, which in turn generates the appropriate forms of the velocity of each individual. These velocity components, in turn, are used to construct a gradient system of first-order ODEs that govern the motion of the swarm. The system is a gradient system because its component is the gradient of the Lyapunov function. Therein lies a major contribution of this paper, and that is, because the Lyapunov function is non-increasing in time, every solution of the system is bounded, yet collision avoidance will occur. This is a depart from the model of Liu et al (2009) which requires an unbounded attraction-repulsion function to guarantee collision avoidance. As explained later in some detail in subsection 5.3, our deterministic system will always have a sufficiently large value of the Lyapunov function at the initial state. This ensures sufficient control efforts for collision avoidance. In other words, during collision avoidance, it is not the Lyapunov function that increases in time, but rather its instantaneous rate of change, with respect to time, that increases in absolute value. This corresponds to sufficient control efforts required for collision avoidance. Another major contribution of this paper is the precise use of three parameters – the convergence, coupling and cohesive parameters – to predict the behavior of the swarm, and to allow for an isotropic and a reciprocal swarm model, or anisotropic and non-reciprocal swarm model, a generality missing in other Lagrangian models discussed above.

Our approach is a result of a development of a Lyapunov-based robot control technique that was proposed by Stonier Stonier (1990), whose work is an application of the Lyapunov method to qualitative differential games that involve dynamical systems subject to control by one or more players [Leitmann and Skowronski (1977), Getz and Leitmann (1979), Stonier (1983)]. Using these differential games concepts, Stonier constructs Lyapunov-like functions that provide nonlinear controllers for collision-avoidance between robot arms, and between robot arms and stationary objects. The main advantage of this *global potential approach*, as classified by Lee (2004), is the ease at which it can be used to extract control laws based on velocity or acceleration. Stonier's work was later expanded and improved by Vanualailai et al (1995, 1998). Their paper was the basis of further improvements by Ha and Shim (2001). In addition, Vanualailai et al (2008) applied their method to the point stabilization of nonholonomic vehicles. Further, Sharma et al (2009, 2010) applied the method to a flock of nonholonomic vehicles .

In this paper, we show that our approach, which utilizes three parameters, not only captures the basic feature of aggregation, cohesion and stability of a swarm, but also exhibits more complex dynamics such as random walks, and self-organized oscillatory motions.

We begin in the next section by re-calling the Direct Method of Lyapunov and the definition of the gradient system.

2 The Lyapunov Function and Gradient Systems

Here, we briefly recall some of the important Lyapunov stability concepts, and properties of gradient systems. The book by Hirsch et al (2004) is a good reference material.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the Euclidean norm $\| \cdot \|$. Let $X = (x_1, x_2, \cdots, x_n)$ denote an element of $\mathbb{R}^n$. Consider an autonomous nonlinear system

$$\dot{X} = F(X), \quad X(t_0) =: X_0, \quad t_0 \geq 0, \tag{1}$$

where $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be smooth enough to guarantee the existence, uniqueness and continuous dependence of solutions $X(t) = X(t; t_0, X_0)$ of (1) in Ω, an open set in $\mathbb{R}^n$.

For the purpose of considering stability concept in the sense of Lyapunov, we assume there is a point $X^* \in \mathbb{R}^n$ such that $F(X^*) \equiv \mathbf{0}$. Then $X(t) \equiv X^*$ is trivially a solution of (1) through $X^* \in \Omega$ for all $t \geq t_0$. We call X^* an *equilibrium point* of system (1)

The equilibrium point X^* of (1) is *stable* if, for each $\epsilon > 0$ and $t_0 \geq 0$, there is a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|X_0 - X^*\| < \delta$ implies $\|X(t) - X^*\| < \epsilon$ for all $t \geq t_0$. The equilibrium point X^* of (1) is said to be *asymptotically stable* if it is stable and there exists $\delta(t_0) > 0$ such that $\|X_0 - X^*\| < \delta$ implies $\lim_{t\to\infty} \|X(t) - X^*\| = 0$.

Lyapunov's Direct Method (also called the Second Method of Lyapunov) is summarized in the following theorem, where $\mathbb{R}^+ := [0, \infty)$:

Theorem 1 (Lyapunov Theorem) *Let X^* be an equilibrium point of (1) and let $V : \Omega \to \mathbb{R}^+$ be a C^1 function defined on some neighborhood Ω of X^* such that (i) $V(X^*) = 0$, (ii) $V(X) > 0$ for $X \in \Omega \setminus \{X^*\}$ and (iii) $\dot{V}_{(1)}(X) \leq 0$ for all $X \in \Omega$. Then X^* is stable. If (iii) is replaced by (iii)' $\dot{V}_{(1)}(X) < 0$ for all $X \in \Omega \setminus \{X^*\}$, then X^* is asymptotically stable.*

The well-known LaSalle's Invariance Principle gives an alternative asymptotic stability condition:

Theorem 2 (LaSalle's Invariance Principle) *Let $V : \Omega \subset \mathbb{R}^n \to \mathbb{R}^+$, $\Omega_c = \{X \in \Omega : V(X) \leq c\}$, and suppose $\dot{V}_{(1)}(X) \leq 0$ on Ω_c. Let $E = \{X \in \Omega_c : \dot{V}_{(1)}(X) = 0\}$. Then every solution of (1) tends to the largest invariant set in E as $t \to \infty$. In particular, if E contains no invariant set other than $\{X^*\}$, then X^* is asymptotically stable.*

We refer to V in Theorem 1 and Theorem 2 as a Lyapunov function for system (1). Its *gradient* is the vector field

$$\nabla V = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n} \right).$$

A *gradient system* on $\mathbb{R}^n$ is a system of differential equations of the form

$$\dot{X} = -\nabla L(X), \tag{2}$$

where L is a Lyapunov function for system (2), and $\dot{L}_{(2)}(X^*) = 0$ if and only if X^* is an equilibrium point of system (2). That is, the critical points of L are the equilibrium points of the system. Moreover, as discussed in Hirsch et al (2004), if a critical point is an isolated minimum of L, then this point is an asymptotically stable equilibrium point of system (2).

3 A Two-Dimensional Swarm Model

A general swarm model, formulated by Mogilner et al (2003), considers a swarm of n individuals being viewed in a coordinate system moving with the velocity of the swarm's centroid. We shall follow this formulation in this paper, but with a divergent approach to the construction of the attraction-repulsion function.

We confine ourselves to constructing a 2-dimensional version of the model, as it will be a simple matter to extend it to 3-dimensional.

At time $t \geq 0$, let $(x_i(t), y_i(t))$, $i = 1, 2, \ldots, n$, be the planar position of the ith individual, which we shall define as a point mass residing in a disk of radius $r_i > 0$,

$$b_i = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 - x_i)^2 + (z_2 - y_i)^2 \leq r_i^2\}. \tag{3}$$

The disk, incidently, is described in Mogilner et al (2003) as a *bin*, and in Gazi and Passino (2004b) as a *private or safety area* of each individual. Also, as discussed in Gazi and Passino (2004a), there are some good reasons why individuals in a swarm could be considered as a point mass; for instance, when considering some organisms such as bacteria.

Let us define the *centroid of the swarm* as

$$(x_c, y_c) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k \right).$$

At time $t \geq 0$, let $(v_i(t), w_i(t) := (x_i'(t), y_i'(t))$ be its instantaneous velocity, which we will need to show that is relative to the swarm centroid.

Using the above notations, we have thus a system of first-order ODEs for the ith individual, assuming the initial condition at $t = t_0 \geq 0$:

$$\left. \begin{array}{r} x_i'(t) = v_i(t) \\ y_i'(t) = w_i(t) \\ x_{i0} := x_i(t_0), y_{i0} := y_i(t_0). \end{array} \right\} \tag{4}$$

Suppressing t, we let $\mathbf{x}_i = (x_i, y_i,) \in \mathbb{R}^2$ and $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^{2n}$ be our state vectors. Also, let

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \underbrace{(x_{10}, y_{10}, \ldots, x_{n0}, y_{n0})}_{2n \text{ terms}}.$$

If $\mathbf{f}_i(\mathbf{x}) := (v_i, w_i) \in \mathbf{R}^2$ and $\mathbf{V}(\mathbf{x}) := (\mathbf{f}_1(\mathbf{x}), \ldots, \mathbf{f}_n(\mathbf{x})) \in \mathbb{R}^{2n}$, then our swarm system of n individuals is

$$\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x}), \quad \mathbf{x}_0 = \mathbf{x}(t_0). \tag{5}$$

An equilibrium point of system (5) for which (4) is the ith component will be denoted $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*) \in \mathbb{R}^{2n}$.

We will use the following two terms from Mogilner et al (2003):

1. A *cohesive* group is a group in which the distances between individuals are bounded from above (members of a cohesive group tend to stay together and avoid dispersing).
2. A *well-spaced* group is a group which does not collapse into a tight cluster, i.e., where some minimal bin size exists such that each bin contains at most one individual. Moreover, the size of such a bin is independent of the number of individuals in a group.

4 A Lyapunov Function with Attraction and Repulsion Components

4.1 Attraction to the Centroid

We can ensure that individuals are attracted to each other and also form a cohesive group by having a measurement of the distance from the ith individual to the swarm centroid. This is the concept behind *flock centering*, which is one of the well-known three heuristic flocking rules of Reynolds (1987). The rule stipulates that individuals stay close to nearest flock mates. It is therefore a form of attraction between individuals. Centering necessitates a measurement of the distance from the the ith individual to the swarm centroid. Thus, we define the following function:

$$R_i(\mathbf{x}) \quad := \quad \frac{1}{2}\left[\left(x_i - \frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 + \left(y_i - \frac{1}{n}\sum_{i=1}^{n} y_i\right)^2\right].$$

It will be part of a Lyapunov function for system (4), and as we shall see later, its role is to ensure that ith individual is attracted to the swarm centroid.

4.2 Inter-individual Collision Avoidance

The short-range repulsion between individuals necessitates a measurement of the distance between the ith and the jth individuals, $j \neq i \in \mathbb{N}$. With the definition (3) of the ith individual in mind, we define the following function for this purpose:

$$R_{ij}(\mathbf{x}) := \frac{1}{2}\left[(x_i - x_j)^2 + (y_i - y_j)^2 - (r_i + r_j)^2\right].$$

It will also be part of the same Lyapunov function we mentioned above.

5 Swarming in the Absence of Obstacles

We first consider the scenario where there are no obstacles in the environment. We formally construct the Lyapunov function and then discuss its form and its relationship to swarming.

5.1 Lyapunov Function

Let there be real numbers $\gamma_i > 0$, $\beta_{ij} > 0$, and define, for $i, j = 1, \ldots, n$,

$$L_i(\mathbf{x}) = \gamma_i R_i(\mathbf{x}) + \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}(\mathbf{x})}.$$

Consider as a tentative Lyapunov function for system (5),

$$L(\mathbf{x}) = \sum_{i=1}^{n} L_i(\mathbf{x}_i).$$

It is clear that L is continuous and locally positive definite on the domain

$$D_1(L) := \left\{\mathbf{x} \in \mathbb{R}^{2n} : R_{ij}(\mathbf{x}) > 0, i, j \in \mathbb{N}\right\}.$$

This means that if $\mathbf{x}^*$ is an equilibrium point of system (5) for which L is a Lyapunov function, then $L(\mathbf{x}) > 0$ for all $\mathbf{x} \in D_1(L) \setminus \{\mathbf{x}^*\}$ and $L(\mathbf{x}^*) = 0$, with $\mathbf{x}^* \in D_1(L)$.

The time-derivative of L along a solution of system (5) is the dot product of the gradient of L,

$$\nabla L = \left(\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial y_1}, \cdots, \frac{\partial L}{\partial x_n}, \frac{\partial L}{\partial y_n} \right),$$

and the time-derivative of the state vector $\mathbf{x} = (x_1, y_1, \ldots, x_n, y_n)$. That is,

$$\dot{L}_{(4)}(\mathbf{x}) = \nabla L(\mathbf{x}) \cdot \dot{\mathbf{x}} = \sum_{i=1}^{n} \left(\dot{R}_i(\mathbf{x}) + \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij}}{R_{ij}(\mathbf{x})} \dot{R}_i(\mathbf{x}) - \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}^2(\mathbf{x})} \dot{R}_{ij}(\mathbf{x}) \right),$$

where

$$\begin{aligned} \sum_{i=1}^{n} \dot{R}_i(\mathbf{x}) \quad &= \quad \sum_{i=1}^{n} \left[\left(x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left(x_m - \frac{1}{n} \sum_{k=1}^{n} x_k \right) \right] x_i' \\ &\quad + \sum_{i=1}^{n} \left[\left(y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - \frac{1}{n} \sum_{m=1}^{n} \left(y_m - \frac{1}{n} \sum_{k=1}^{n} y_k \right) \right] y_i', \end{aligned}$$

and

$$\sum_{i=1}^{n} \sum_{\substack{j=1,\\ j\neq i}}^{n} \dot{R}_{ij}(\mathbf{x}) = 2 \sum_{i=1}^{n} \sum_{\substack{j=1,\\ j\neq i}}^{n} (x_i - x_j) x_i' + 2 \sum_{i=1}^{n} \sum_{\substack{j=1,\\ j\neq i}}^{n} (x_i - x_j) y_i'.$$

Noting that $\sum_{m=1}^{n} \left(u_m - \frac{1}{n} \sum_{k=1}^{n} u_k \right) = 0$ for any $u_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, we simplify the former expression to

$$\sum_{i=1}^{n} \dot{R}_i(\mathbf{x}) = \sum_{i=1}^{n} \left[\left(x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) x_i' + \left(y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) y_i' \right].$$

Now, collecting terms with x_i' and y_i', and substituting $x_i' = v_i$ and $y_i' = w_i$ from system (4), we have

$$\begin{aligned} \dot{L}_{(5)}(\mathbf{x}) \quad &= \quad \sum_{i=1}^{n} \left\{ \left(\gamma_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij}}{R_{ij}(\mathbf{x})} \right) \left(x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - 2 \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}^2(\mathbf{x})} (x_i - x_j) \right\} \dot{x}_i \\ &\quad + \sum_{i=1}^{n} \left\{ \left(\gamma_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij}}{R_{ij}(\mathbf{x})} \right) \left(y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - 2 \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}^2(\mathbf{x})} (y_i - y_j) \right\} \dot{y}_i \\ &= \quad \sum_{i=1}^{n} \left[\frac{\partial L}{\partial x_i} \cdot \dot{x}_i + \frac{\partial L}{\partial y_i} \cdot \dot{y}_i \right] = \sum_{i=1}^{n} \left[\frac{\partial L}{\partial x_i} \cdot v_i + \frac{\partial L}{\partial y_i} \cdot w_i \right]. \end{aligned}$$

Let there be real numbers $\alpha_i^1 > 0$ and $\alpha_i^2 > 0$ such that

$$v_i = -\alpha_i^1 \frac{\partial L}{\partial x_i}, \quad \text{and} \quad w_i = -\alpha_i^2 \frac{\partial L}{\partial y_i}.$$

Then

$$\dot{L}_{(5)}(\mathbf{x}) = -\sum_{i=1}^{n} \left[\alpha_i^1 \left(\frac{\partial L}{\partial x_i} \right)^2 + \alpha_i^2 \left(\frac{\partial L}{\partial y_i} \right)^2 \right] = -\sum_{i=1}^{n} \left[\frac{v_i^2}{\alpha_i^1} + \frac{w_i^2}{\alpha_i^2} \right] \leq 0,$$

for all $\mathbf{x} \in D_1$.

For the ith individual, system (4) therefore becomes

$$\left.\begin{aligned} x_i'(t) = v_i(t) = -\alpha_i^1 \frac{\partial L}{\partial x_i}, \\ y_i'(t) = w_i(t) = -\alpha_i^2 \frac{\partial L}{\partial y_i}, \\ x_{i0} = x_i(t_0), y_{i0} = y_i(t_0), \quad t_0 \geq 0, \end{aligned}\right\} \tag{6}$$

where

$$\frac{\partial L}{\partial x_i} = \left(\gamma_i + \sum_{\substack{j=1, \\ j \neq i}}^{n} \frac{\beta_{ij}}{R_{ij}(\mathbf{x})} \right) \left(x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right) - 2 \sum_{\substack{j=1, \\ j \neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}^2(\mathbf{x})} (x_i - x_j),$$

and

$$\frac{\partial L}{\partial y_i} = \left(\gamma_i + \sum_{\substack{j=1, \\ j \neq i}}^{n} \frac{\beta_{ij}}{R_{ij}(\mathbf{x})} \right) \left(y_i - \frac{1}{n} \sum_{k=1}^{n} y_k \right) - 2 \sum_{\substack{j=1, \\ j \neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}^2(\mathbf{x})} (y_i - y_j).$$

Define the $n \times n$ diagonal matrix

$$A = \operatorname{diag}(\underbrace{\alpha_1^1, \alpha_1^2, \ldots, \alpha_n^1, \alpha_n^2}_{2n \text{ elements}}).$$

Then our system (5) becomes the gradient system

$$\dot{\mathbf{x}} = -A\,(\nabla L(\mathbf{x})), \quad \mathbf{x}_0 := \mathbf{x}(t_0), \quad t_0 \geq 0, \tag{7}$$

the ith term of which is given by (6).

We now establish that $L(\mathbf{x})$ is indeed Lyapunov function for system (7), and that it provides a stable equilibrium point for the system.

Theorem 3 *The function $L(\mathbf{x})$, $\mathbf{x} \in D_1(L)$, is a Lyapunov function for system (7), a stable equilibrium point of which is*

$$\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*) := \underbrace{\left(\frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k, \cdots, \frac{1}{n} \sum_{k=1}^{n} x_k, \frac{1}{n} \sum_{k=1}^{n} y_k \right)}_{2n \text{ terms}} \in D_1(L).$$

Proof. By the Chain Rule

$$\dot{L}_{(7)}(\mathbf{x}) = \nabla L(\mathbf{x}) \cdot \dot{\mathbf{x}} = \nabla L(\mathbf{x}) \cdot [-A\,(\nabla L(\mathbf{x})].$$

Since A is an $n \times n$ diagonal matrix with real-valued entries, $\alpha_i^s > 0$, $i = 1, \ldots, n$ and $s = 1, 2$, if $\lambda := \max\{\alpha_i^s; i = 1, \ldots, n, s = 1, 2\}$, then

$$\dot{L}_{(7)}(\mathbf{x}) \leq -\lambda \left|\nabla L(\mathbf{x})\right|^2 \leq 0,$$

showing that $L(\mathbf{x})$, with $\mathbf{x} \in D_1(L)$, is a Lyapunov function for system (7). In particular $\dot{L}(\mathbf{x}) = 0$ if and only $\nabla L(\mathbf{x}) = 0$. Since $\dot{L}_{(7)}(\mathbf{x}) = 0$ at $\mathbf{x} = \mathbf{x}^*$, it follows easily that $\mathbf{x}^*$ is a stable equilibrium point of system (7) and is in $D_1(L)$.

As discussed in Gazi and Passino (2004a), swarming in nature normally occurs in a distributed fashion; there is no leader and each individual decides independently its direction of motion. Our model captures this since system (6) gives the equations of motion of each individual, and does not depend on an external variable (such as a command from a leader or another agent), but only on the position of the individual itself and its observation of the positions (or relative positions) of the other individuals. Moreover, the individuals do not have to know the global Lyapunov function $L(\mathbf{x})$. Instead, they only know the local or their internal Lyapunov function $L_i(\mathbf{x})$.

5.2 Insight into the form of the Lyapunov Function and Cohesiveness

Let us now discuss the idea behind the construction of our Lyapunov function

$$L(\mathbf{x}) = \sum_{i=1}^{n} L_i(\mathbf{x}) = \sum_{i=1}^{n} \left(\gamma_i R_i(\mathbf{x}) + \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}(\mathbf{x})} \right).$$

At large distances between the ith and the jth individuals, the ratio,

$$\sum_{i=1}^{n} \sum_{\substack{j=1,\\ j\neq i}}^{n} \frac{\beta_{ij} R_i(\mathbf{x})}{R_{ij}(\mathbf{x})}, \tag{8}$$

is negligible, and the attraction term $\sum_{i=1}^{n} \gamma_i R_i(\mathbf{x})$ dominates. Thus, the long-range attraction requirement in a swarm model is met, and $\sum_{i=1}^{n} \gamma_i R_i$ acts as the *attraction function*. Indeed, since $\sum_{i=1}^{n} \gamma_i R_i(\mathbf{x})$ is allowed to be zero at $\mathbf{x} = \mathbf{x}^*$ where L is also zero, and $dL/dt \leq 0$ for all $t \geq 0$ along every solution of (7), each individual is attracted to the centroid, and therefore the swarm system (7) maintains centering and hence cohesiveness at all times. Indeed, Theorem 3 proves the cohesiveness of the swarm since stability means that for each $\epsilon > 0$ and $t_0 \geq 0$, there is a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|\mathbf{x}(t_0) - \mathbf{x}^*\| < \delta$ implies $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for all $t \geq t_0$, with $\mathbf{x}, \mathbf{x}^* \in D_1(\mathbf{x})$; this boundedness of solution for all time $t \geq t_0$ implies that distances between individuals are bounded from above at all times. Note that the parameter $\gamma_i > 0$ can be considered as a measurement of the strength of attraction between an individual i and the swarm centroid, and hence between each other. The smaller the parameter is, the weaker the attraction between the members is; hence, γ_i can be considered a *coupling parameter*.

Consider the situation where any two individuals i and j approach each other. In this case, R_{ij} decreases and the ratio increases, with $\beta_{ij} > 0$ acting as a *cohesion parameter* that is a measurement of the strength of interaction between the individuals. Now, because, with respect to time $t \geq 0$, we have that $dL/dt \leq 0$ along a trajectory of system (7), and L is a positive definite function, L *cannot* increase in $t \geq t_0 \geq 0$. Hence, for every initial condition $\mathbf{x}(t_0) \in D_1(\mathbf{x})$, the ratio *cannot* be unbounded in t. However, at the initial time $t_0 \geq 0$, large values of $L(\mathbf{x}(t_0))$ – and hence large controls efforts – could be required for collision-avoidance and cohesion. This is so because the ratio (8) is large for small arguments. Fuelled with this large value of L which can only decrease in t, any change in the value of the ratio (8) could only correspond to either an increase or decrease in $|dL/dt|$. Analogously, $|dL/dt|$ is the rate of dissipation of energy from the system in absolute value. Thus, if two individuals approach each other, the rate of energy dissipation from system (7), in absolute value, gets larger. This increased dissipation of energy along a trajectory of system (7) could only be directed towards a stable equilibrium point, such as $\mathbf{x}^*$, where $R_{ij} \neq 0$, away from any potential collision between individuals i and j. In other words, we cannot have a situation where $R_{ij} = 0$ along any system trajectory that starts in $D_1(\mathbf{x})$. Hence, the ratio (8) acts as an *obstacle-avoidance function* and system trajectories originating at $\mathbf{x}(t_0) \in D_1(\mathbf{x})$ remain in $D_1(\mathbf{x})$ for all time $t \geq t_0 \geq 0$. In turn, this means that the individuals in a swarm cannot collapse onto themselves.

If the two parameters γ_i and β_{ij} are the same for all individuals, then we have an isotropic and a reciprocal swarm model. If they differs between at least two individual, then the model is anisotropic and non-reciprocal.

Finally, we note that we have used two other parameters, $\alpha_i^1 > 0$ and $\alpha_i^2 > 0$, $i = 1, \ldots, n$ in system (7). Because the parameters are a measure of the instantaneous velocities, we name them *convergence parameters*. The larger the convergence parameters, the quicker the movements of the individuals towards and about the centroid.

5.3 Constant Arrangement About the Centroid

Theorem 3 shows that the swarm members can converge to a constant arrangement about the swarm centroid. Indeed, if

$$S := \{\mathbf{x} \in \mathbb{R}^{2n} : \frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial y_i} = 0, i = 1, \ldots, n\},$$

which is the set of all equilibrium points of system (7), and

$$E := \{\mathbf{x} \in D_1(\mathbf{x}) : \dot{L}(\mathbf{x}) = 0\}, \ \ \dot{L} = \dot{L}_{(5)} = \dot{L}_{(7)}$$

then it follows easily that $S = E$ since

$$\dot{L}(\mathbf{x}) = -\sum_{i=1}^{n} \left[\alpha_i^1 \left(\frac{\partial L}{\partial x_i} \right)^2 + \alpha_i^2 \left(\frac{\partial L}{\partial y_i} \right)^2 \right] = 0,$$

if and only if

$$\frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial y_i} = 0, \quad i = 1, \cdots, n.$$

Hence, by LaSalle's Invariance Principle (Theorem 2), the equilibrium points in S, which include $\mathbf{x}^*$, are attractive.

5.4 Size and Density of the Swarm

Given that a member i of the swarm resides in a disk defined in (3), with radius r_i, we can follow the argument by Gazi and Passino (2004a) to estimate the size and density of the swarm in a stable arrangement, but without using their assumption that the swarm members had to be squeezed cohesively as closely as possible in an area (a disk) of radius r, since Theorem 3 already provides this cohesiveness. Indeed, since Theorem 3 establishes the stability of system (7) in $D_1(\mathbf{x})$, there is no collision between members in $D_1(\mathbf{x})$. Accordingly, between two members i and j,

$$\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| > (r_i + r_j), \ \ \mathbf{x}_i = (x_i, y_i),$$

for all time $t \geq t_0 \geq 0$. Now, the safety areas are disjoint, so the total area occupied by the swarm is $\pi \sum_{i=1}^{n} r_i^2$. By Theorem 3, for each $\epsilon > 0$ and $t_0 \geq 0$, there is a $\delta = \delta(t_0, \epsilon) > 0$ such that $\|\mathbf{x}(t_0) - \mathbf{x}^*\| < \delta$ implies $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for all $t \geq t_0$, with $\mathbf{x}, \mathbf{x}^* \in D_1(\mathbf{x})$. In such a stable arrangement, where all the solutions of (7) are bounded above by $\epsilon > 0$, we can therefore find a disk of radius, say, $r = r(\epsilon)$, around $\mathbf{x}^*$ such that

$$\pi r^2(\epsilon) \geq \pi \sum_{i=1}^{n} r_i^2.$$

From this we get

$$r_{\min} := \sqrt{\sum_{i=1}^{n} r_i^2},$$

a lower bound on the radius of the smallest circle which can enclose all the individuals. It is clear that the swarm size will will scale with the size of the individual.

If we define the density of the swarm as the number of individuals per unit area, and let it be ρ, then it is simple to see that ρ is upper bounded, with

$$\rho \leq \frac{n}{\pi \sum_{i=1}^{n} r_i^2}.$$

Hence, the swarm cannot become arbitrarily dense.

6 Computer Simulations

Extensive computer simulations show that for a sufficiently large number of individuals the proposed model (7) generates four types of swarming-like behaviors. They are (1) the cruise formation (linear or nonlinear) reminiscent of a cruising and leaderless school of fish, a moving herd of cattle or elephants with a leader (leader-following), (2) random walks similar to the swarming behavior of fruit flies *Drosophila melanogaster*, (3) constant arrangements requiring individuals to aggregate and stop, as in fruiting body formation by bacteria, and (4) circular motions reminiscent of the behavior of a school of fish when threatened by a predator.

Table 1 summarizes the emergent behaviors as we modify the three parameters.

6.1 Examples of Type A and Type B Arrangements

6.1.1 Straight Line Formation

Our first diagram shows an example of Type A arrangement. In Figure 1 (a), randomly-positioned 30 individuals, each with bin 10, at the initial time of $t = 0$ are shown. As time evolves, they cluster around the centroid and cruise along a straight line as a well-spaced cohesive group as shown in Figure 1 (b). The path of the centroid is shown thick.

Table 1: Parameters produce different types of emergent behaviors for sufficiently large populations.

Type	Convergence parameter, $\alpha_i^s > 0$, $s = 1, 2$; $i \in \mathbb{N}$	Coupling parameter, $\gamma_i > 0$, $i \in \mathbb{N}$	Cohesion parameter, $\beta_{ij} > 0$, $i, j \in \mathbb{N}$, $i \neq j$	Some emergent arrangement about centroid
A	same α_i^s for all s, i, or random α_i^s	same γ_i for all i	same β_{ij} for all i, j	– coherent compact cluster cruising in a straight line; – constant arrangement.
B	same α_i^s for all s, i	random γ_i	same β_{ij} for all i, j	– coherent compact cluster cruising in a nonlinear fashion, with leader(s) possibly emerging; – circular motion.
C	same α_i^s for all s, i or, random α_i^s	same γ_i for all i	random β_{ij}	– Lévy-like random walk;
D	random α_i^s	random γ_i	same β_{ij} for all i, j	– same as in B
F	same α_i^s for all s, i	random γ_i	random β_{ij}	– same as in C
G	random α_i^s	random γ_i	random β_{ij}	– any of the above

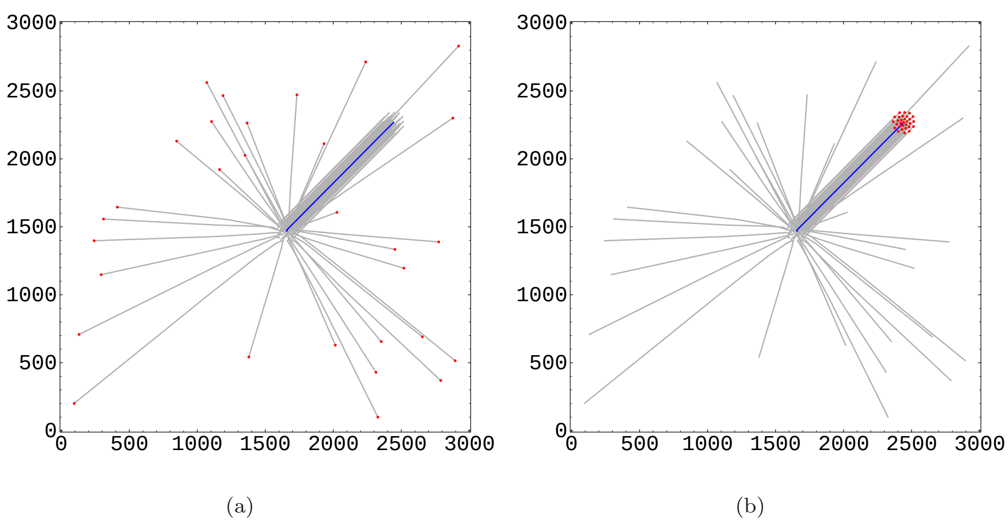


Figure 1: Cruise. Example of Type A arrangement, where $\alpha_i^s = 1$, $\gamma_i = 2$ and $\beta_{ij} = 50$. The path of the centroid is shown thick. The swarm is cruising non-stop as a well-spaced cohesive group in a stable formation.

6.1.2 Constant Arrangement about Centroid

In 2006, Sozinova and colleagues developed a three-dimensional model of myxobacterial fuiting-body formation, in which myxobacterial cells, when sensing starvation, change their movement pattern from outward spreading to inward concentration and form aggregates nucleated by a stationary traffic jam or nonsymmetric initial aggregates [Sozinova et al (2006)].

In our second example (Figure 2), the members ($n = 70$, bin 20) converge to a constant arrangement about the centroid, reminiscent of the shape of the base level of such bacterial swarm formation.

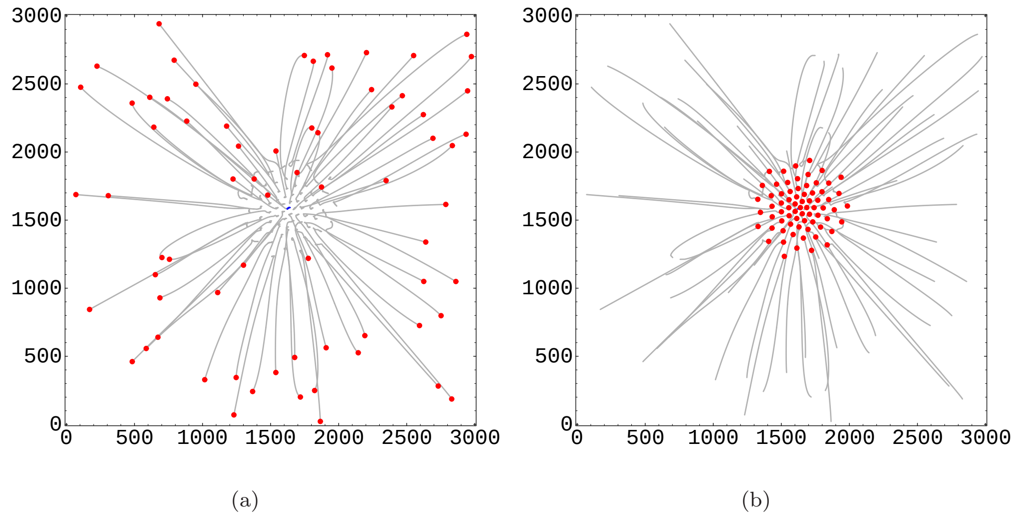


Figure 2: Balanced forces. Example of constant arrangement about centroid (Type A), where $\alpha_i^s = 0.1$, $\gamma_i = 0.2$ and $\beta_{ij} = 50$. Part (a) and (b) show the initial and final positions of the individuals, respectively. The centroid remains stationary.

6.1.3 Leader-following Behavior

The example shown in Figure 3, with $n = 30$ individuals and bin 10, shows a nonlinear path taken by the swarm, with an emergent leader.

6.2 Examples of Type C Arrangement

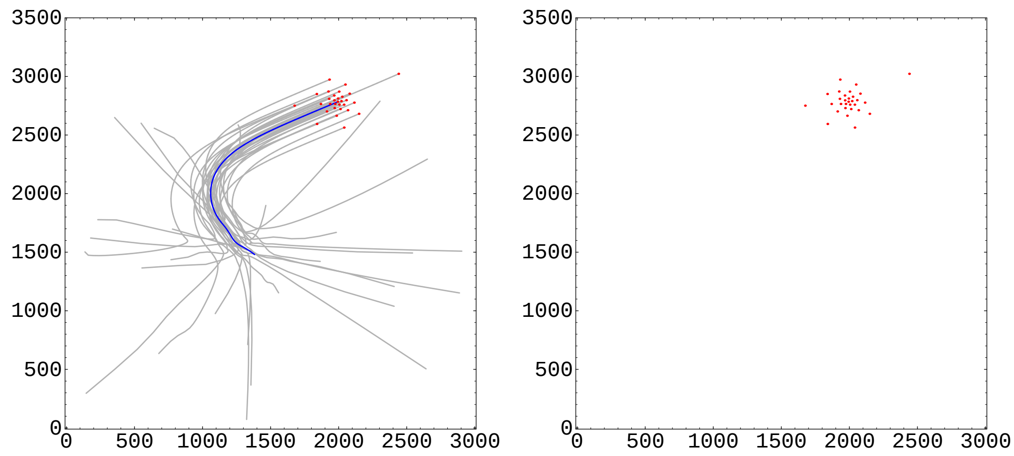
6.2.1 Random Walks

In 2006, Majkut modelled the flight paths of fruit flies *Drosophila melanogaster*, which utilize scent to locate food sources in their vicinity [Majkut (2006)]. Fruit fly flight is characterized by a series of straight segments interrupted by rapid changes in horizontal heading known as *saccades*. Majkut used *Lévy flights* to model the foraging behavior of fruit flies. Levy flights are a class of continuous time random walks, which are often found in biological behavior and are prevalent in foraging.

Our diagram in Figure 4 shows a Lévy-like random walk, with $n = 30$ and bin 10.

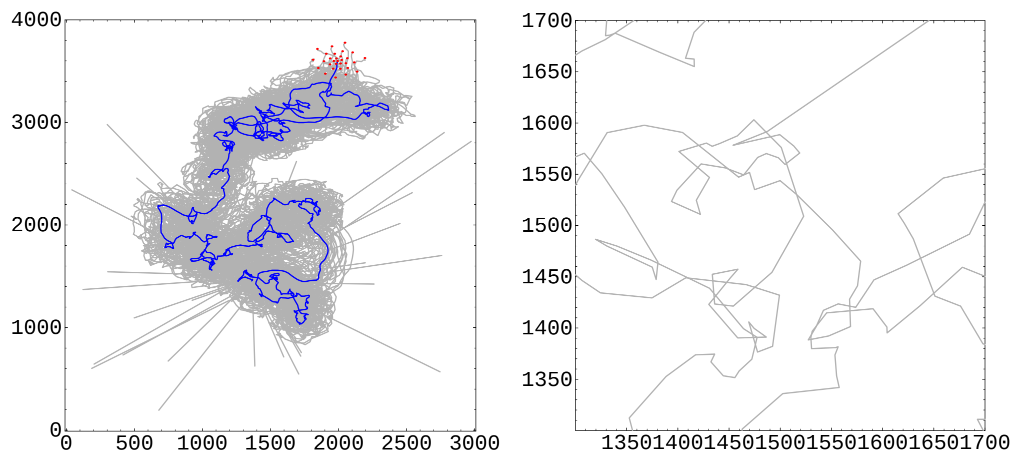
6.2.2 Circular Motion from Random Walks

Our second example of Type C arrangement shows the formation of a circular motion out from a random walk (Figure 5).



(a) (b)

Figure 3: Leader of the pack. Example of Type B arrangement, where $\alpha_i^s = 1$, γ_i is randomized between between 0.01 and 1, and $\beta_{ij} = 100$. The path of the centroid is shown thick. The swarm is cohesive throughout. Part (b), without paths drawn, clearly shows a leader and those that trail behind.



(a) (b)

Figure 4: Random walks. Example of Type C arrangement, where $\alpha_i^s = 5$, $\gamma_i = 1$ and β_{ij} is randomize between and including 200 and 500. The path of the centroid is shown thick in (a). The swarm is cohesive throughout. In (b), the path of an individual is magnified, showing a saccade-like flight path.

7 Conclusion

Recent work on swarm modelling, especially by Edelstein-Keshet (2001), Mogilner and Edelstein-Keshet (1999), Mogilner et al (2003), and Gazi and Passino (2003, 2004b,a) shows

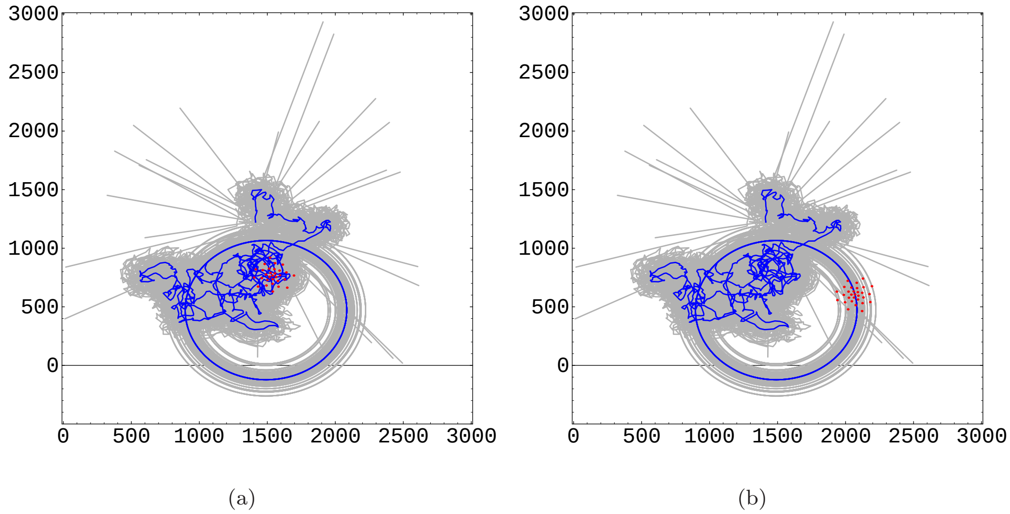


Figure 5: From randomness to order. Example of Type C arrangement, where $\alpha_i^s = 5$, $\gamma_i = 2$ and β_{ij} is randomize between 200 and 500. The path of the centroid is shown thick. The swarm is cohesive throughout. Initially, in (a), there is randomness. At some certain time, order in the form of a circular motion began (b). The motion is clockwise.

that an element of the swarming phenomenon is a long-range attraction and a short-range repulsion between individuals in the swarm. The Lagrangian approach is a means to do this. This study also supports this heuristic argument with a novel technique to construct a Lagragian model. Utilizing the Lyapunov method, we create a gradient system that is stable, implying the congregation of individuals about their centroid to form cohesive and well-spaced swarms.

Our model shares the disadvantage of other Lagrangian models which require every individual to know (or sense) the (relative) position of all the other individuals. Obviously such models are not scalable. Nonetheless, it has several characteristics that make it stand out from other swarm models: (1) It is a distributed system that not only captures the basic feature of aggregation, cohesion and stability of a swarm, but also exhibits more complex dynamics such as random walks and self-organized oscillatory motions via the use of only three parameters; the convergence, coupling and cohesion parameters; (2) It is general enough to be either an isotropic and a reciprocal swarm model, or anisotropic and non-reciprocal swarm model by manipulating the coupling and cohesion parameters appropriately; (3) Finally, the results may be applicable to distributed robotic systems, or considered for the control of heterogenous robotic swarms by creating, for instance, a kinematic model of an individual robot in a swarm and constructing its instantaneous velocity along the method expounded in this article. The recent work by the authors and colleagues in Sharma et al (2009, 2010), which include fixed obstacles, is in this direction, and will be further developed in light of the new results in this paper.

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